@Chen Siyi

November 30, 2020

Note9 Fourier Series

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> Let's look at the series methods from a wider view

1 Inner Product Space

1.1 Vector Space

A **vector space** is a set V , equipped with a rule for addition of any two vectors and for scalar multiplication of a vector by a scalar.

The **addition** + must satisfy the following axioms:

1. The set V is closed under addition: For any two vectors **v** and **w** of V, the sum $v+w$ is also in V.
2. Additionis commutative: For all $v, w \in V$, $v+w=w+v$.
3. Additionis associative: For all $v, w, y \in V$, $(v+w)+y=v+(w+y)$.
4. There is an additive identity: There exists $0_V \in V$ such that $v+0_V=0_V+v=v$ for all $v \in V$.
5. Every element has an additive inverse: for every $v \in V$, there exists a vector $y \in V$ such that $v+y=y+v=0_V$.

The **scalar multiplication** must satisfy the following axioms:

1. The set V is closed under scalar multiplication: For any vector **v** in V and any scalar $\lambda \in \mathbb{R}$, the scalar multiple λv is also in V .
2. For two scalars $a, b \in \mathbb{R}$ or $\mathbb{C}$, we have $a(bv) = (ab)v$ for all vectors $v \in V$.
3. For $1 \in \mathbb{R}$, we have $1v = v$ for all $v \in V$.

And finally, **scalar multiplication distributes over addition**:

1. $\lambda(v + w) = \lambda(v) + \lambda(w)$ fo rall v,w∈V and all $\lambda \in \mathbb{R}$ or $\mathbb{C}$.
2. $(a + b)v = av + bv$ for all vectors $v \in V$ and all scalars $a, b \in \mathbb{R}$ or $\mathbb{C}$.

1.2 Inner Product Space

An **inner product** on a **vector space** V is a function

$$V \times V \longrightarrow X$$

X can be $\mathbb{C}$ or $\mathbb{R}$. It must satisfy the following axioms:

1. Symmetry: $\langle f, g \rangle = \overline{\langle g, f \rangle}$ for all vectors $f, g \in V$;
2. Linearity in each factor: $\langle (f + g), h \rangle = \langle f, h \rangle + \langle g, h \rangle$ for all vectors f,g,h ∈ V and $\langle \lambda f, g \rangle = \overline{\lambda} \langle f, g \rangle$ for all $\lambda \in \mathbb{C}$ and all $f, g \in V$.
3. Positive Definiteness: $\langle f, f \rangle \geq 0$ for all $f \in V$ with $\langle f, f \rangle = 0$ if only if $f = 0_V$.

A vector space V together with a choice of an inner product $(V, < \cdot, \cdot >)$ is called an **inner product space**.

> As the most familiar example, $V = \mathbb{R}^n$, $< \cdot, \cdot > = \cdot$

1.3 Norm and Orthogonality

> As soon as you define an inner product space, you can define norm and orthogonality

Norm of a vector $\mathbf{v}$ is defined as

$$||\mathbf{v}|| = \sqrt{< \mathbf{v}, \mathbf{v} >}$$

Which can be interpreted as the length defined under that inner product.

If the norm of $\mathbf{v} = 1$, it is said to be **normed** or **normalized**.

And there are some **fundamental theorems**:

1. **Pythagoras's Theorem:**

   Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space and $z = x + y \in V$, where $\langle x, y \rangle = 0$. Then

   $$||z||^2 = ||x||^2 + ||y||^2$$

2. **Bessel's Inequality:**

   Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space and $\{e_k\}$, $k \in I \subset V, I \subset N$, be an orthonormal system in V. Then, for any $v \in V$,

   $$\sum_{k \in I} |\langle e_k, v \rangle|^2 \leq ||v||^2$$

Two vectors $\mathbf{v}$ and $\mathbf{w}$ are **orthogonal** if

$$< \mathbf{v}, \mathbf{w} >= 0$$

A system of vectors is an **orthonormal system** if all the vectors are **normed**, and are **orthogonal** to each other.

> Let's see a really straight-forward way to find orthonormal systems!
>
> As long as initially you have a system, just keep using **orthogonal projection**...

1.4 Gram-Schmidt Orthonormalization

Used to construct an orthonormal system from an existing system.

$$w_1 := \frac{v_1}{\|v_1\|}$$

$$w_2 := \frac{v_2 - \langle w_1, v_2 \rangle w_1}{\|v_2 - \langle w_1, v_2 \rangle w_1\|}$$

$$w_k := \frac{v_k - \sum_{j=1}^{k-1} \langle w_j, v_k \rangle w_j}{\left\|v_k - \sum_{j=1}^{k-1} \langle w_j, v_k \rangle w_j\right\|}, \quad k = 2, \ldots, n$$

> **A Tricky Question:**
>
> By the way, does any system returns an orthonormal system of the same size as before?

1.5 Orthonormal Basis and Completeness

A system of vectors $\mathcal{B}$ is an **orthonormal basis** of the vector space V if it is an **orthonormal system**, and $\mathbf{span}\{\mathcal{B}\} = V$.

> **A Tricky Question:**
>
> Can we always construct an **orthonormal basis** of V using the **Gram-Schmidt Orthonormalization** approach from an existing basis of V?

A theorem but not the definition: An **inner product space** $(V, < \cdot, \cdot >)$ is **complete** iff every maximal orthonormal system is an orthonormal basis.

> Intuitively you can interpret as...
>
> Any vector in V can be represented "precisely" (by linear combinitions) using the maximal orthonormal system as long as it is a basis.

> Otherwise, even you have a maximal orthonormal system, this system can't be used to represent a vector precisely!

Multiple inner products can be defined for a vector space. For example, let $C([a,b])$ be the vector space:

$$\|f\|_2 := \sqrt{\int_a^b |f(x)|^2 dx}$$

$$\|f\|_1 := \int_a^b |f(x)| dx$$

$$\|f\|_\infty := \sup_{x\in[a,b]} |f(x)|$$

And a sequence is said to **converge uniformly** if $\| f_n \| \infty \to 0$, **converge in the mean** if $\| f_n \|_1 \to 0$,and **converge in the mean square** if $\| f_n \|_2 \to 0$.

But not all inner products make sure the inner product space is complete.

For example, $(C([a,b]), \|\cdot\|_2)$ is incomplete. $(L^2([a,b]), \|\cdot\|_2)$ is complete. Where,

$$L^2([a,b]) := \left\{ f : [a,b] \to \mathbb{C} : \int_a^b |f(x)|^2 dx < \infty \right\}$$

2 Approximation of "Vectors"

We now "say" the **least square estimation** is our **best approximation**.

Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space, $v \in V$ and $B = e_k$ an orthonormal system in V . We seek to approximate a vector v using a linear combination of the first $N \in \mathbb{N}$ elements (or as many as you can...) of the orthonormal system:

$$v \approx \sum_{i=1}^N \lambda_i e_i, \qquad \lambda_1, \ldots, \lambda_N \in \mathbb{F}$$

Our approximation is the best when the following "error" is minimized:

$$\left\| v - \sum_{i=1}^N \lambda_i e_i \right\|^2$$

> **Fourier analysis** is acting exactly in the same way to find an estimation...
>
> And we choose the inner product space to be $(L^2([a,b]), \|\cdot\|_2)$. So you will notice, it's generally helpful to represent periodic functions.

> **Which region** $[a,b]$ do we choose?
>
> **Which orthonormal system** do we choose? Four different choices. Keep in mind it is affected by the region you choose.

> **A Tricky Question:**
>
> Recall **The Method of Frobenius**, do you find any similarities in the methods?

3 Fourier Analysis

3.1 Basic Definitions about "Measure"

- A set $\Omega \subset \mathbb{R}$ is said to have
  - **measure less than** ε if...
  - **measure zero** if...
- A property is said to hold **almost everywhere** (abbreviated by a.e.) on a subset $D \subset \mathbb{R}$ if...
- The set $\Omega = \mathbb{Q}$ of rational numbers has measure zero.

(One of) Our next goal:

Find some (Fourier)**series** f' equal to a **function** f almost everywhere within certain region.

3.2 The Fourier-Euler Basis

If you choose $[a,b] = [-\pi, \pi]$, then since the below ***trigonometric polynomials*** forms an orthonormal basis of $L^2([-\pi, \pi])$, we can choose it as the **Fourier-Euler Basis**.

$$\mathscr{B}_{\mathcal{G}} = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos(nx), \frac{1}{\sqrt{\pi}} \sin(nx) \right\}_{n=1}^{\infty}$$

1. You can just use the first several terms to form a ***Fourier-Euler expansion*** of any $f \in L^2([-\pi, \pi])$ by projecting onto the first sevral vectors in this basis.
2. You can also form a ***Fourier Series*** f' which is a linear combination of "every" vectors in the basis, and equals to $f \in L^2([-\pi, \pi])$ **almost every where**. f' is gained by projecting onto each vector in this basis.

If you choose $[a,b] = [0, L]$, then the **Fourier-Euler Basis** can be

$$\mathscr{B}_1 := \left\{ \frac{1}{\sqrt{L}}, \sqrt{\frac{2}{L}} \cos\left(\frac{2\pi n x}{L}\right), \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi n x}{L}\right) \right\}_{n=1}^{\infty}$$

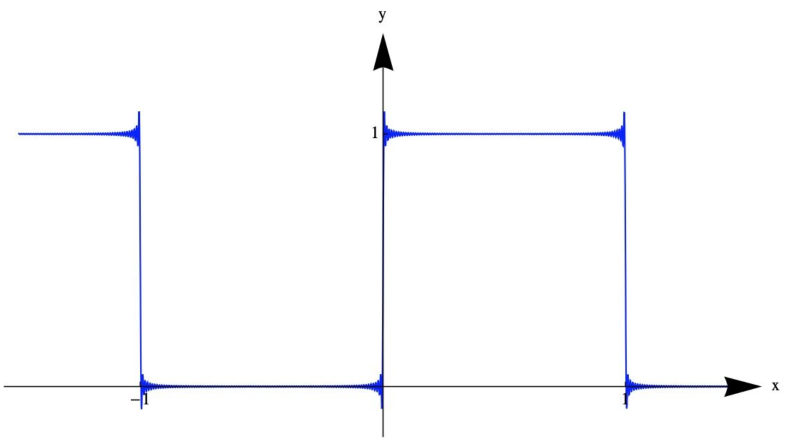
> **Exercise:**
>
> If possible, uniquely expand $f(x) = x^2, 0 < x < 2\pi$ in a Fourier series if (a) the period is 2π, (b) the period is not specified.

> **Exercise:**
>
> Using the results of the last exercise to prove
>
> $$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

3.2.1 The Gibbs Phenomenon

A **Fourier series** may not and does not need to **converge uniformly**. The occurrence of these **peaks** of f' at the **discontinuities** of f is known as the **Gibbs phenomenon**.

> Recall "uniform convergence".



3.2.2 Convergence of Fourier Series(Dirichlet's rule)

1. On any sub-interval $[a', b'] \subset [a, b]$ with $a' > a$, $b' < b$ on which f is continuous, the **Fourier series converges uniformly** towards f .
2. At any point $x \in [a, b]$, we have the pointwise limit: when $N \to \infty$

$$S_N(x) \longrightarrow \frac{\lim_{y \nearrow x} f(y) + \lim_{y \searrow x} f(y)}{2}$$

> Actually the second point needs certain conditions () to hold, but we don't discuss them in VV286.

3.3 The Fourier-Cosine Basis

Cosine functions are even.

- For even functions, we can just use Cosine functions to expand.
- For functions which is not even, we can just use Cosine functions to find its even part.

If you choose $[a, b] = [0, L]$, then the **Fourier-Cosine Basis** can be

$$\mathscr{B}_2 := \left\{ \frac{1}{\sqrt{L}}, \sqrt{\frac{2}{L}} \cos\left(\frac{\pi n x}{L}\right) \right\}_{n=1}^{\infty}$$

3.4 The Fourier-Sine Basis

Since functions are odd.

- For odd functions, we can just use Sine functions to expand.
- For functions which is not even, we can just use Cosine functions to find its odd part.

If you choose $[a, b] = [0, L]$, then the **Fourier-Sine Basis** can be

$$\mathscr{B}_3 := \left\{ \sqrt{\frac{2}{L}} \sin\left(\frac{\pi n x}{L}\right) \right\}_{n=1}^{\infty}$$

> **A Tricky Question:**
>
> For an arbitrary $f \in L^2([a, b])$, if you expand it using the the Fourier-Cosine Basis and the Fourier-Sine Basis separately with ∞ terms, and obtain series f_1 and f_2, would $f_1 + f_2$ and f becomes equal almost every where?
>
> It's interesting to see knowledge links to each other, does this remind you of any fundamental theorems you have seen before?

> **Exercise:**
>
> Explain why any function $F(x)$ is a sum of an even function and an odd function in just one way.
>
> Explain why any odd function $f(x)$ would has its Fourier cosine series be 0.

> **A Tricky Question:**
>
> Can the following question be solved?
>
> Expand $f(x) = sinx$, $0 < x < \pi$, in a Fourier cosine series.

> **Exercise:**
>
> Expand $f(x) = x$, $0 < x < 2$, in a half range (a) sine series, (b) cosine series.
>
> **Hint:** Can you first "extand" f a little bit?

> Just a reminder before the last part, $L^2([a,b]) := \left\{ f : [a,b] \to \mathbb{C} : \int_a^b |f(x)|^2 dx < \infty \right\}$ is defined to containning **complex valued functions of a real variable**, not the same as **complex functions**.

3.5 Complex Fourier-Euler Basis

If you choose $[a,b] = [-\pi, \pi]$, then the following would also be a basis of $L^2([a,b])$, simply because you can change basis with the Fourier-Euler Basis which we discussed above.

$$\mathscr{B}_{\mathcal{G}} = \left\{ \frac{1}{\sqrt{2\pi}} e^{inx} \right\}_{n=-\infty}^{\infty}$$

So you can also do orthogonal projection onto each vector of this basis to find an approximation. No matter whether f is real valued or not, since just the λ_i might be complex numbers.

4 Additional Practice

> **Exercise:**
>
> Find a Fourier series for $f(x) = cos\alpha x$, $-\pi < x < \pi$, where $\alpha \neq 0, \pm 1, \pm 2, \pm 3, \ldots$ .... .
>
> **Hint:** Can you first "extand" f a little bit?

Exercise:

Prove that

$$\sin x = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{(2\pi)^2}\right)\left(1-\frac{x^2}{(3\pi)^2}\right)\cdots$$

Hint:

Use the results from the previous exercise.