@Chen Siyi

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Note8 Bessel Equations

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> Let's apply the Method of Frobenius to solve Bessel equations.
>
> And analyze the solutions (Bessel functions).

1 Bessel Equations of Order v

$$x^2y'' + xy' + \left(x^2 - v^2\right) y = 0$$

> Having a regular singular point at 0.
>
> The Method of Frobenius can be applied.

1.1 Find the Indical and Recurrence Equations

Choose the ***Frobenius ansatz***

$$x(t) = t^r \sum_{k=0}^{\infty} a_k t^k \qquad a_0 \neq 0$$

Besides,

$$xp(x) = 1, \qquad p_0 = 1$$

$$x^2q(x) = x^2 - v^2, \qquad q_0 = -v^2, \quad q_2 = 1$$

Setting

$$F(x) := x(x-1) + p_0 x + q_0 = x^2 - v^2$$

We get the ***indicial equation*** and ***recurrence equations***

$$\begin{aligned} F(r) &= r^2 - v^2 = 0 \\ a_m F(r+m) &= -\sum_{k=0}^{m-1} \left(q_{m-k} + (r+k)p_{m-k}\right) a_k, \quad m \geq 1 \end{aligned}$$

Which gives us

$$\begin{aligned} & r^2 - v^2 = 0 \\ & a_1\left((r+1)^2 - v^2\right) = 0 \\ & a_m = -\frac{a_{m-2}}{(m+r+v)(m+r-v)}, \quad m \geq 2 \end{aligned}$$

It obviously turns out $r_1 = v$ and $r_2 = -v$.

> From the result in class we know if $r_1 - r_2 = 2v \notin \mathbb{N}$, two independent solutions would be found easily.
>
> And if $r_1 - r_2 = 2v \in \mathbb{N}$, we may use the special technique.
>
> ***However***, we will see actually for ***Bessel Equations***, the condition is slightly ***less strict***:
>
> ***If $v \notin \mathbb{N}$, then r_1 and r_2 give two independent solutions.***

1.2 Find the First Independent Solution

1.2.1 Find the First Independent Solution with the Larger r_1

With the ***LARGER*** $r_1 = v$, we have

$$\begin{aligned} & a_1\left((v+1)^2 - v^2\right) = 0 \\ & a_m = -\frac{a_{m-2}}{(m+2v)m}, \quad m \geq 2 \end{aligned}$$

So $a_1 = a_3 = a_5 = \cdots = 0$ and

$$a_{2k} = \frac{(-1)^k a_0}{2^{2k} k!(1+v)(2+v)\cdots(k+v)}$$

> **Question:**
>
> Notice v may not be an integer. Don't write as factories.
>
> Then how do you simpliy this solution?

1.2.2 The Bessel Function of the First Kind

Recall ***Euler Gamma function***'s property:

$$\Gamma(s+1) = s\Gamma(s)$$

So it gives

$$(1+v)(2+v)\cdots(k+v) = \frac{\Gamma(k+1+v)}{\Gamma(1+v)}$$

And by setting $a_0 = \frac{2^{-v}}{\Gamma(1+v)}$, we will have the first independent solution be ***the Bessel function of the first kind of order*** v

$$J_v(x) = \left(\frac{x}{2}\right)^v \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+1+v)} \left(\frac{x}{2}\right)^{2k}$$

> **Question:**
>
> Which region of x does $J_v(x)$ defined?

Take $v = 1$ as example, we have

$$J_1(x) = \frac{x}{2} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}(k+1)!k!}$$

1.3 Find the Second Independent Solution ($v \notin \mathbb{N}$)

Starting from if $2v$ ***is not an integer***, with the ***SMAllER*** $r_2 = -v$, we have

$$\begin{aligned} a_1((v-1)^2 - v^2) &= 0, \quad a_1(2v-1) = 0 \\ a_m &= -\frac{a_{m-2}}{(m-2v)m}, \quad m \geq 2 \end{aligned}$$

We have $a_1 = a_3 = a_5 = \cdots = 0$ and

$$a_{2k} = \frac{(-1)^k a_0}{2^{2k} k!(1-v)(2-v)\cdots(n-v)}$$

Similarly,

$$(1-v)(2-v)\cdots(k-v) = \frac{\Gamma(k+1-v)}{\Gamma(1-v)}$$

And by setting $a_0 = \frac{2^{-v}}{\Gamma(1+v)}$, the second independent solution will be ***the Bessel function of the first kind of negative order*** $-v$

$$J_{-v}(x) = \left(\frac{x}{2}\right)^{-v} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+1-v)} \left(\frac{x}{2}\right)^{2k}$$

Then the ***general solution*** is

$$y(x) = C_1 J_v(x) + C_2 J_{-v}(x)$$

But actually, If $2v$ ***is an odd integer***, which means v ***is not an integer***, the above results also holds.

And the combined conclusion is if v is not an integer, the above results will hold.

1.4 Find the Second Independent Solution ($v \in \mathbb{N}$)

1.4.1 Reduction of Order

Set $y_2(x) = c(x) \cdot J_\nu(x)$, then

$$\begin{aligned} & x^2 y_2'' + x y_2' + \left(x^2 - \nu^2\right) y_2 = 0 \\ \Rightarrow & x^2 \left(c''(x) J_\nu(x) + 2c'(x) J_\nu'(x) + c(x) J_\nu''(x)\right) \\ & + x\left(c'(x) J_\nu(x) + c(x) J_\nu(x)\right) + \left(x^2 - \nu^2\right) c(x) \cdot J_\nu(x) = 0 \\ \Rightarrow & x^2 J_\nu(x) c''(x) + \left(2x^2 J_\nu'(x) + x J_\nu(x)\right) c'(x) = 0 \\ \Rightarrow & \ln|c'(x)| = \left(-2\ln|J_\nu(x)| - \ln|x|\right) \\ \Rightarrow & c'(x) = \frac{1}{x \cdot J_\nu^2(x)} \\ \Rightarrow & c(x) = \int \frac{dx}{x \cdot J_\nu^2(x)} \end{aligned}$$

So a second independent solution is given as

$$y_2(x) = J_\nu(x) \int \frac{dx}{x \cdot J_\nu^2(x)}$$

1.4.2 The Other Method

$$x_2(t) = \frac{\partial}{\partial r}\left(t^r \sum_{k=0}^{\infty} a_k(r) t^k\right)\Bigg|_{r=r_2} = c \cdot x_1(t) \ln t + t^{r_2} \sum_{k=0}^{\infty} a_k'(r_2) t^k$$

$$\frac{a_{2k}'(r)}{a_{2k}(r)} = \frac{d}{dr} \ln|a_{2k}(r)|$$

> **Practice:**
>
> Using 5 minites to try solving out the second solution by yourself.
>
> Do you find any problems?

Instead of computing $a_{2k}'(r_2)$, let's find these new constants in another way. Assume

$$y_2(x) = a J_v(x) \ln x + x^{-v}\left[\sum_{k=0}^{\infty} c_k x^k\right], \quad x > 0$$

Computing $y_2\prime, y_2\prime\prime(x)$, substituting in the original Bessel Equation, and make use of $J_v(x)$ is a solution(as we have done by reduction of order), we can obtain all the constants $a, c_0, c_1, \ldots$

Let's try with the Bessel Equation of order 1.

$$y_2(x) = aJ_1(x)\ln x + x^{-1}\left[\sum_{k=0}^{\infty} c_k x^k\right], \quad x > 0$$

Substituting back and since $J_1(x)$ is a solution, we can simplify the equation to be

$$2axJ_1'(x) - c_1 + \sum_{k=2}^{\infty}(k^2 - 2k)c_k x^{k-1} + \sum_{k=0}^{\infty} c_k x^{k+1} = 0$$

Substituting for $J_1(x)$ then

$$a\left[\sum_{k=0}^{\infty} \frac{(-1)^k(2k+1)x^{2k+1}}{2^{2k}(k+1)!k!}\right] - c_1 + \sum_{k=0}^{\infty}\left[\left(k^2+2k\right)c_{k+2} + c_k\right]x^{k+1} = 0$$

This first gives us $c_1 = 0$.

Further even powers of the left sum must vanish, so $\left(k^2+2\right)c_{k+2} + c_k$ must vanish for odd k, and then $c_1 = c_3 = \cdots = 0$.

And from setting the coefficients of odd powers as 0, we have

$$\left[(2k+1)^2 - 1\right]c_{2k+2} + c_{2k} = a\frac{(-1)^{k+1}(2k+1)}{2^{2k}(k+1)!k!}, \quad k = 0, 1, 2, 3, \ldots$$

For $k = 0$, we have

$$0 \cdot c_2 + c_0 = -a$$

Now we notice $c_0 = -a$ can be non-zero arbitraty real numbers, and we set $c_0 = 1$ and then $a = -1$. Then

$$\left[(2k+1)^2 - 1\right]c_{2k+2} + c_{2k} = \frac{(-1)^k(2k+1)}{2^{2k}(k+1)!k!}, \quad k = 1, 2, 3, \ldots$$

For $k = 1$, we get

$$\left(3^2 - 1\right)c_4 + c_2 = (-1)3/\left(2^2 \cdot 2!\right)$$

Hence, c_2 can be selected in arbitrary, and then we fix the second independent solution.

In practice, we always choose $c_2 = \frac{1}{2^2}$, and then we would be possible to simplify:

$$c_{2m} = \frac{(-1)^{m+1}\left(H_m + H_{m-1}\right)}{2^{2m}m!(m-1)!}$$

Where $H_m(x) := \sum_{i=1}^{m}\frac{1}{i}$, $H_0 = 0$, is the Harmonic Numbers. So in conclusion we obtain:

$$y_2(x) = -J_1(x)\ln x + \frac{1}{x}\left[1 - \sum_{m=1}^{\infty}\frac{(-1)^m\left(H_m + H_{m-1}\right)}{2^{2m}m!(m-1)!}x^{2m}\right], \quad x > 0$$

1.4.3 The Bessel Function of the Second Kind

Actually the second independent solution of Bessel Equations are written as ***the Bessel function of the second kind of order*** v, which can be some linear combinition of $J_v(x)$ and the second independent solution $y_2(x)$. In our specific case here of order 1, we set ***the Bessel function of the second kind of order 1*** as

$$Y_1(x) = \frac{2}{\pi}[-y_2(x) + (\gamma - \ln 2)J_1(x)]$$

But, in practice, ***the Bessel function of the second kind of order*** v can be found from $J_v(x)$ and $J_{-v}(x)$:

$$Y_v(x) = \frac{J_v(x)\cos \pi v - J_{-v}(x)}{\sin \pi v}$$

And then the ***general solution*** can be written as

$$y(x) = C_1 J_v(x) + C_2 Y_v(x)$$

2 Reduce Differential Equations to Bessel Equation

2.1 $x^2 y'' + xy' - \left(x^2 + v^2\right) y = 0$

> **Exercise:**
>
> Show that the general solution of this equation can be expressed as
>
> $y(x) = C_1 J_v(-ix) + C_2 Y_v(-ix)$

2.2 $x^2 y'' + xy' + \left(a^2 x^2 - v^2\right) y = 0$

> **Exercise:**
>
> Show that the general solution of this equation can be expressed as
>
> $y(x) = C_1 J_v(ax) + C_2 Y_v(ax)$

2.3 $x^2 y'' + axy' + \left(x^2 - v^2\right) y = 0$

> **Exercise:**
>
> Show that the general solution of this equation can be expressed as
>
> $y(x) = x^{\frac{1-a}{2}} \left[C_1 J_n(x) + C_2 Y_n(x)\right]$
>
> **Hint:**
>
> using the substitution $y(x) = x^{\frac{1-a}{2}} z(x)$

2.4 * For thoughts: $y'' - xy = 0$

> ***Exercise:**
>
> Show that the general solution of this equation can be expressed as
>
> $y(x) = C_1 \sqrt{x} J_{\frac{1}{3}} \left(\frac{2}{3} i x^{\frac{3}{2}}\right) + C_2 \sqrt{x} J_{-\frac{1}{3}} \left(\frac{2}{3} i x^{\frac{3}{2}}\right)$
>
> **Hint:**
>
> be careful with $\frac{d^2 y}{dx^2}$