Q1

$$\mathcal{L}(t^a) = \int_0^\infty e^{-st} t^a dt$$

where, if we make the substitution $u = st$ we get:

$$\mathcal{L}(t^a) = \frac{1}{s^{a+1}} \int_0^\infty e^{-u} u^a du = \frac{\Gamma(a+1)}{s^{a+1}}$$

In particular, if a is a natural number n then we get:

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}.$$

which is consistent with our results for t^2 and 1.

Def: $\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx$

$\therefore \Gamma(a+1) = \int_0^\infty e^{-x} x^a du$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Q 2

- Since there are no poles to the right of the Bromwich contour, we find $x(t) = 0$ for $t < 0$

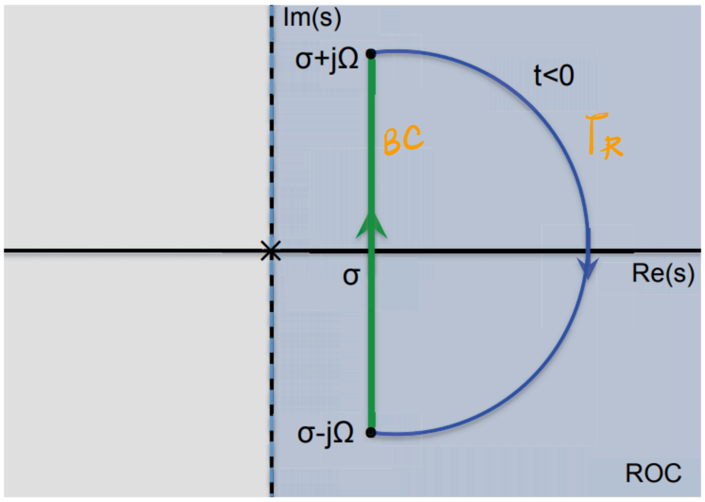
① Use Cauchy's theorem + Jordan's lemma

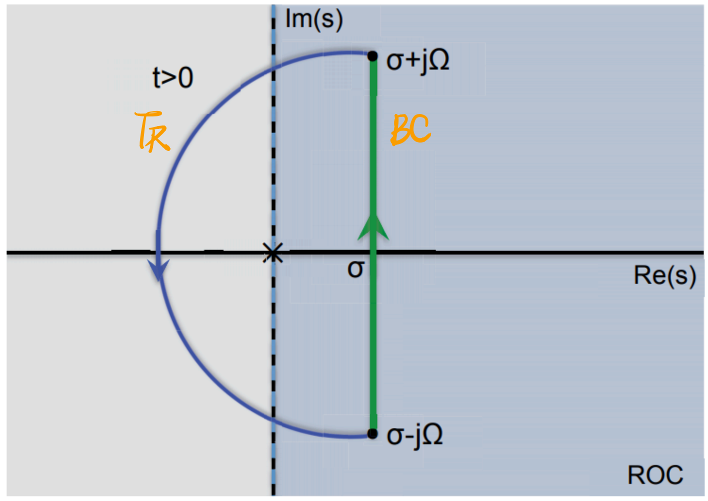
$\int_{T_R} + \int_{BC} = 0 \rightarrow \int_{BC} = 0 \leftarrow \int_{T_R} = 0$

- The simple pole at $s = 0$ is located to the left of the Bromwich contour outside the ROC, of course

① Use Residue theorem + Jordan's lemma

$\int_{T_R} + \int_{BC} = \sum \text{res}$ $\qquad \int_{T_R} = 0$ ☆ how?





- Computing its residue, we find

$$\varphi(s) = sX(s)e^{st} = e^{st} \quad \text{and} \quad \text{Res}[\frac{e^{st}}{s}, 0] = \frac{\varphi(s)}{0!}\Big|_{s=0} = \frac{1}{1} = 1$$

- and the time signal is

$$x(t) = 1 \qquad \text{for } t > 0$$

- Conclusion: $x(t) = u(t)$

☆$_1$ $\text{res}_0 = \lim_{s\to 0} (s \cdot X(s) e^{st}) = 1$

☆$_2$ Jordan's lema: remember to change to

$\sup_{0 \le \theta \le \pi} |g(Re^{i\theta})| \xrightarrow{R\to\infty} |g(z)| \xrightarrow{z\to\infty} 0$

$f(z) = e^{iaz} g(z)$ for $a>0$.

Then $\lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0$

Need to prove: $\lim_{R\to\infty} \int_{T_R} e^{pt} \frac{1}{p} dp$ vanishes.

Here you need to change Γ_R to C_R:

Let $p' = (p-\sigma)(-i) \Rightarrow \quad p = ip' + \sigma$

$$\int_{\Gamma_R} e^{pt} \frac{1}{p} dp = \int_{C_R} e^{(ip'+\sigma)t} \frac{1}{ip'+\sigma} i dp'$$

$$= \int_{C_R} i e^{i(pt)} e^{\sigma t} \frac{1}{ip+\sigma} dp$$

$$= \int_{C_R} e^{itp} \frac{ie^{\sigma t}}{ip+\sigma} dp$$

$\forall$ now, $a \longleftrightarrow t$

$$g(z) \longleftrightarrow \frac{ie^{\sigma t}}{iz+\sigma}$$

$$|g(Re^{i\theta})| = \left| \frac{ie^{\sigma t}}{iRe^{i\theta}+\sigma} \right| \xrightarrow{R \to \infty} 0$$

By Jordan's lemma, the integral vanishes.

You can omit materials starting from this page.

- What happens at $t = 0$? See later.
- Using the inversion formula, we find

$$x(0) = \frac{1}{2\pi j} \int_{s \in \mathrm{Br}} \frac{1}{s} \, ds = \lim_{\substack{\Omega_1 \to \infty \\ \Omega_2 \to \infty}} \int_{\sigma - j\Omega_1}^{\sigma + j\Omega_2} \frac{1}{s} \, ds$$

- By changing the ratio Ω_1 / Ω_2 we can give the integral any value that we want
- Setting $\Omega_1 / \Omega_2 = 1$ (as is usual), the resulting integral is known as a *Cauchy principal value* integral

- With this choice, we have

$$\begin{aligned} x(0) &= \frac{1}{2\pi j} \int_{s \in \mathrm{Br}} \frac{1}{s} \, ds = \frac{1}{2\pi j} \lim_{\Omega \to \infty} \Big[\ln|s| + j \arg(s) \Big]_{s = \sigma - j\Omega}^{\sigma + j\Omega} \\ &= \frac{1}{2\pi j} \cdot 2j \cdot \lim_{\Omega \to \infty} \arctan\left(\frac{\Omega}{\sigma} \right) \\ &= \frac{1}{2\pi j} \cdot 2j \cdot \frac{\pi}{2} = \frac{1}{2} \end{aligned}$$

- For this reason, the Heaviside unit step function is often defined as

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{1}{2} & \text{for } t = 0 \\ 1 & \text{for } t > 0 \end{cases}$$

- The above result can be generalized to a general discontinuous signals
- We have

$$\frac{x(t+0) + x(t-0)}{2} = \frac{1}{2\pi j} \int_{s \in \mathrm{Br}} X(s) e^{st} \, ds$$

Q3

The first step in using Laplace transforms to solve an IVP is to take the transform of every term in the differential equation.

$$\mathcal{L}\{y''\} - 10\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{5t\}$$

Using the appropriate formulas from our **table of Laplace transforms** gives us the following.

$$s^2Y(s) - sy(0) - y'(0) - 10(sY(s) - y(0)) + 9Y(s) = \frac{5}{s^2}$$

also calculated in Q1

$L(t^n) = \frac{n!}{p^{n+1}}$.

Plug in the initial conditions and collect all the terms that have a $Y(s)$ in them.

$$(s^2 - 10s + 9)Y(s) + s - 12 = \frac{5}{s^2}$$

Solve for $Y(s)$.

$$Y(s) = \frac{5}{s^2(s-9)(s-1)} + \frac{12-s}{(s-9)(s-1)}$$

At this point it's convenient to recall just what we're trying to do. We are trying to find the solution, $y(t)$, to an IVP. What we've managed to find at this point is not the solution, but its Laplace transform. So, in order to find the solution all that we need to do is to take the inverse transform.

Before doing that let's notice that in its present form we will have to do partial fractions twice. However, if we combine the two terms up we will only be doing partial fractions once. Not only that, but the denominator for the combined term will be identical to the denominator of the first term. This means that we are going to partial fraction up a term with that denominator no matter what so we might as well make the numerator slightly messier and then just partial fraction once.

This is one of those things where we are apparently making the problem messier, but in the process we are going to save ourselves a fair amount of work!

Combining the two terms gives,

$$Y(s) = \frac{5 + 12s^2 - s^3}{s^2(s-9)(s-1)}$$

The partial fraction decomposition for this transform is,

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \boxed{\frac{C}{s-9}} + \boxed{\frac{D}{s-1}}$$

why? a theorem.

Setting numerators equal gives,

$$5 + 12s^2 - s^3 = As(s-9)(s-1) + B(s-9)(s-1) + Cs^2(s-1) + Ds^2(s-9)$$

Picking appropriate values of s and solving for the constants gives,

$$\begin{array}{llll} s=0 & 5 = 9B & \Rightarrow & B = \frac{5}{9} \\ s=1 & 16 = -8D & \Rightarrow & D = -2 \\ s=9 & 248 = 648C & \Rightarrow & C = \frac{31}{81} \\ s=2 & 45 = -14A + \frac{4345}{81} & \Rightarrow & A = \frac{50}{81} \end{array}$$

⇐ the 1st method to get partial fraction decomposition

Plugging in the constants gives,

$$Y(s) = \frac{\frac{50}{81}}{s} + \frac{\frac{5}{9}}{s^2} + \frac{\frac{31}{81}}{s-9} - \frac{2}{s-1}$$

Finally taking the inverse transform gives us the solution to the IVP.

$$y(t) = \frac{50}{81} + \frac{5}{9}t + \frac{31}{81}\mathbf{e}^{9t} - 2\mathbf{e}^{t}$$

recall: $1 \overset{L}{\Rightarrow} \frac{1}{p}$

$e^{at}g(t) \overset{L}{\Rightarrow} (Lg)(p-a)$