@Chen Siyi

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Note6 Laplace Transform

The (Unilateral) Laplace Transform

Definition of (Unilateral) Laplace Transform

Let $f:[0,\infty)\to\mathbb{R}$ be a continuous function such that

$$\sup_{t\in[0,\infty)} e^{-pt}|f(t)|<\infty \quad \text{for some } p\geq 0$$

Then the function $F:(\beta,\infty)\to\mathbb{R}$,

$$F(p):=(\mathscr{L}f)(p):=\int_0^\infty e^{-pt}f(t)dt$$

is called the ***(Unilateral) Laplace transform*** of f.

> **A Reminder**
>
> Why should the "sup" condition be satisfied for the (Unilateral) Laplace transform?
>
> For any f, will there always exist a region of p where $F(p)$ is defined?

And the ***ROC(radius of convergence)*** is the region for p, where $F(p)$ converges. So the (Unilateral) Laplace Transform $(\mathscr{L}f)(p)$ is defined for such p.

> **A Tricky Question**
>
> Can the ROC of the (Unilateral) Laplace Transform of a function f be in the form $(-\infty, \beta)$?

Definition of (Bilateral) Laplace Transform

The bilateral Laplace transform is defined as

$$(\widetilde{\mathscr{L}} f)(p) := \int_{-\infty}^{\infty} f(t) e^{-pt} \, dt$$

Of course, we have

$$\mathscr{L} f = \widetilde{\mathscr{L}} (Hf)$$

The following questions may help you understand better the relationship between these two types of Laplace Transform.

> **A Tricky Question**
>
> What should the value of p satisfy for the (Bilateral) Laplace Transform of f to exist?

> **A Tricky Question**
>
> If for a funstion f, the Unilateral Laplace Transform of f is defined in some region of p, will there always be a Bilateral Laplace Transform for certain region of p?

> **A Tricky Question**
>
> Can the ROC of the (Bilateral) Laplace Transform of a function f be in the form $(-\infty, \beta)$, (α, β)?

*Our main focus here (VV286) is the **(Unilateral) Laplace Transform**.

> **Question**
>
> Find the Laplace transform of a function of the form t^a with $a > -1$.
>
> Hint:
>
> - We focus on the **Unilateral** Laplace transform witout stating out.
> - Considering the gamma function, what's its definition?

> **Answer**

Properties of (Unilateral) Laplace Transform

$f(t)$	$(\mathscr{L}f)(p)$	Comment
$H(t-a)$	e^{-ap}/p	$a, p > 0$
$g(t-a)H(t-a)$	$e^{-ap}(\mathscr{L}g)(p)$	$a > 0$
$e^{at}g(t)$	$(\mathscr{L}g)(p-a)$	$a \in \mathbb{R}$
$g(at)$	$\frac{1}{a}(\mathscr{L}g)\left(\frac{p}{a}\right)$	$a > 0$
$g^{(n)}(t)$	$p^n(\mathscr{L}g)(p) - p^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	$n \in \mathbb{N}$
$(-t)^n g(t)$	$(\mathscr{L}g)^{(n)}(p)$	$n \in \mathbb{N}$

- $\mathscr{L}(f * g) = (\mathscr{L}f) \cdot (\mathscr{L}g)$
- $\mathscr{L}(\int_0^t f(\tau)d\tau) = \frac{1}{p}\mathscr{L}(f(t))$

The Inverse of the Laplace Transform

The laplace transform can be extended to the ***complex plane***, by using the ***analytic continuation*** where we set $F(z)$ by replacing p with z in $F(p)$ for $Rez > \beta$. (Why?)

For $z = p + iq$, where $p > \beta$ we have

$$|F(z)| \leq \int_0^\infty \left|f(t)e^{-zt}\right| dt = \int_0^\infty |f(t)|e^{-pt} dt$$
$$\leq M \int_0^\infty e^{-(p-\beta)t} dt = \frac{M}{p-\beta}$$

1. $F(z)$ exists.
2. $F(z)$ is holomorphic. Since the integral converges absolutely, and the function $g(z) = e^{zt}$ is complex differentiable for any $t \in [0, \infty)$. (Recall what is F).

The Bromwich Integral

Definition

Let $\Omega \subset \mathbb{C}$ be an open set, $\beta \in \mathbb{R}$ and $F : \Omega \to \mathbb{C}$ analytic for all $z \in \mathbb{C}$ with $Rez \geq \beta$. Then the ***Bromwich integral*** of F is defined as

$$(\mathscr{M}F)(t) = \frac{1}{2\pi i} \int_{C^*} e^{pt} F(p) dp$$

where $\mathcal{C} = z \in C : Rez = \beta$ is the ***Bromwich contour***, oriented ***positively*** if the contour is closed on the left (i.e., the line is traversed in the direction of positive imaginary part.)

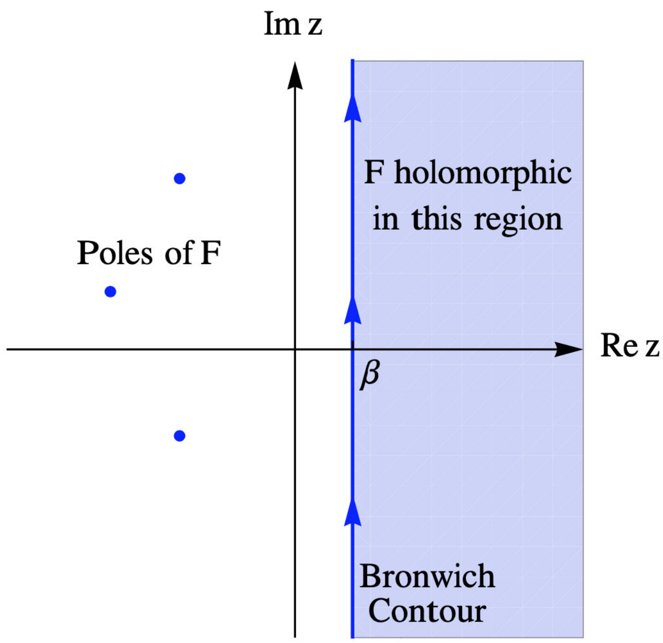
Often, the integral is written

$$(\mathscr{M}F)(t) = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} e^{pt} F(p) dp$$

An example for the ***Bromwich integral*** of a function F is given below.

> **A Tricky Question**
>
> Does it matter which β you choose?
>
> --*Yes and no*.

> **A Reminder**
>
> Where $F(p)$ is a ***Unilateral Laplace Transform***, and the Bromwich contour must be located within the ROC. So it will always be holomorphic in the right part of the Bromwich integral.
>
> Such $f(t)$ is ***causal*** because there are no poles to the right of the Bromwich contour.



Evaluation

Summary

1. The ***region of*** t, i.e. whether $t < 0$ or $t > 0$, decides which ***semi-circle*** you would like to use for integral.

> Why?

2. The ***positions of poles***, decide which ***theorem*** to use for integral.

> Which theorems? How do you decide?

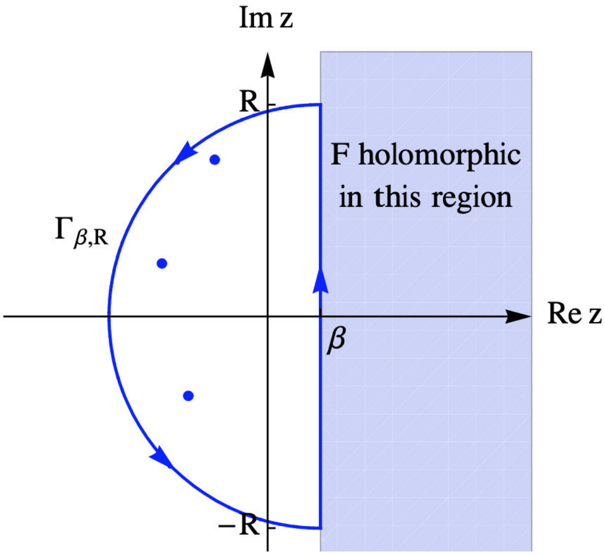
Concrete Analysis

- $t > 0$

$$\int_{\Gamma_{\beta,R}} e^{pt}F(p)dp = ie^{\beta t}\int_{C_R} e^{itp}F(\beta + ip)dp$$

Jordan's lema shows the integral would ***vanish*** as $R \to \infty$.

Use the ***Residue Theorem*** for this specific case.

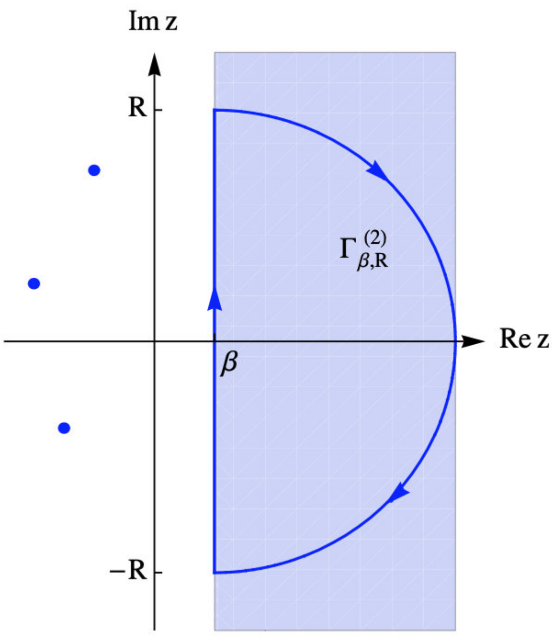


- $t < 0$

$$\int_{\Gamma^{(2)}_{\beta,R}} e^{pt}F(p)dp = ie^{\beta t}\int_{C_R} e^{i|t|p}F(\beta - ip)dp$$

  Jordan's lema shows the integral would ***vanish*** as $R \to \infty$.

  Use ***Cauthy's Integral Theorem*** for this specific case.



> **A Tricky Question**
>
> "Whether the integral along the two semi-circles vanishes", is it related to the poles of F?

The Mellin Inversion Formula

If f is continuous on $[0, \infty)$, continuously differentiable on $(0, \infty)$ and satisfies $\sup_{t\in[0,\infty)} e^{-\beta t}|f(t)| < \infty$ for some $\beta > 0$, then

$$f(s) = [\mathscr{M}(\mathscr{L}f)](s) \quad \text{for all } s \in [0, \infty)$$

which is called the ***Mellin inversion formula*** for the Laplace transform.

> **Question**
>
> **Let $F(p) = 1/p$ be the Laplace transform of a time signal $f(t)$ with the half-plane Re(s) > 0 as its ROC. Find $f(t)$ using the *Mellin inversion formula*.**
>
> Hint:
>
> - We focus on the Unilateral Laplace transform witout stating out.
> - The Bromwich contour must be located within the ROC
> - *What happens to the point $t = 0$?

> **Answer**

Solving Differential Equations with Laplace Transform

Overall Idea

1. Apply the ***Laplace transform*** to both sides of the ODE/IVP.
2. ***Solve*** for the Laplace transform Y.
3. Find ***inverse Laplace transform***, which is the solution. You can try (in mixture of) several ways:
    1. Decomposite $Y(p)$ into partial fractions.
    2. Use the table of pairs, and also properties.

3. Use the mellin inversion formula.

> **Question**
>
> Solve the IVP:
>
> $$y'' - 10y' + 9y = 5t, \quad y(0) = -1 \quad y'(0) = 2$$

> **Answer**

Using Green's Function and Convolution

- A Convolution Product for the Laplace Transform:

$$\mathscr{L}(f * g) = (\mathscr{L}f) \cdot (\mathscr{L}g)$$

- A linear, second order, inhomogeneous ODE with constant coefficients:

$$ay'' + by' + cy = f(x), \quad y(0) = y_0, y'(0) = y_1$$

1. Apply the ***Laplace transform*** to both sides of the ODE/IVP.
2. ***Solve*** for the Laplace transform Y.

$$Y = (\mathscr{L}f)(p) \cdot \frac{1}{ap^2 + bp + c} + \frac{ay_0 p + by_0 + ay_1}{ap^2 + bp + c}$$

3. Find the ***Green's Function*** *g*(*x*) such that $(\mathscr{L}g)(p) = \frac{1}{ap^2+bp+c}$
4. Use transform table and apply convolution to find ***inverse Laplace transform***, which is the solution.

> **Question**
>
> Solve the IVP:
>
> $$y'' + y = \begin{cases} \cos t, 0 \leqslant t \leqslant \pi/2 \\ 0, \pi/2 \leqslant t < \infty \end{cases} = \cos t \cdot H\left(\frac{\pi}{2} - t\right), y(0) = 3, y'(0) = -1$$

> **Answer**

Impulses and the Delta Function