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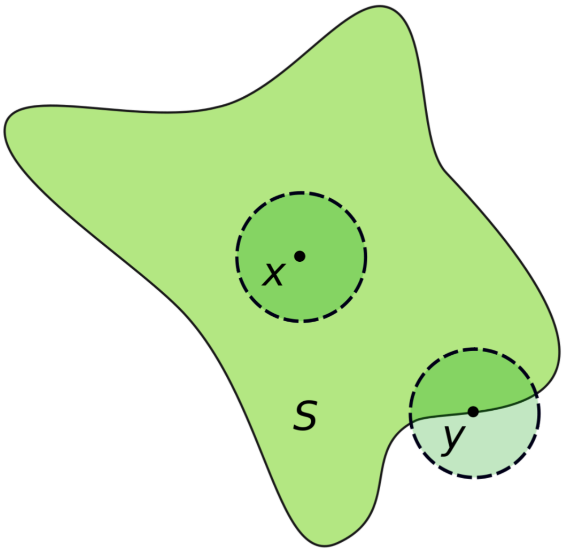
Note4 Complex Analysis

Points in the Complex Plane



- For a given $z \in \mathbb{C}$ and $\varepsilon > 0$, the set

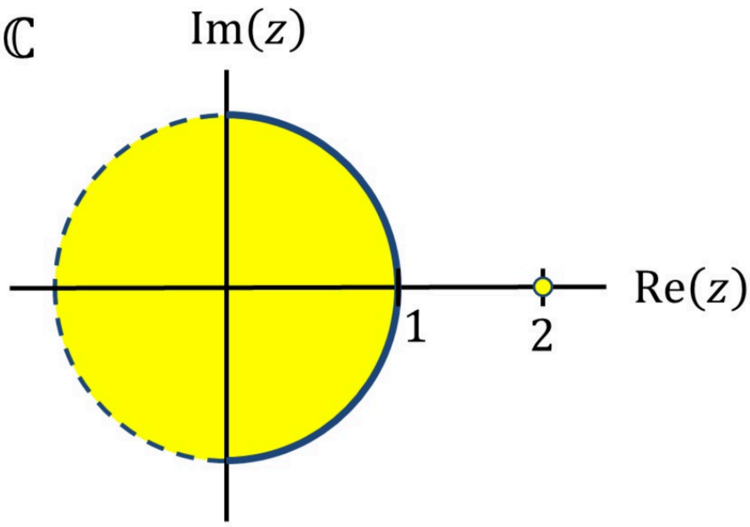
  $B_\varepsilon(z) = \{w \in \mathbb{C} \mid |w - z| < \varepsilon\}$,

  is called an $\varepsilon - \text{neighborhood}$ of z;

  $B_\varepsilon(z) = \{w \in \mathbb{C} \mid 0 < |w - z| < \varepsilon\}$,

  is called an $\varepsilon - \text{deleted neighborhood}$ of z.
- A point z_0 is an ***interior point*** of set $S \subset \mathbb{C}$ if there is some ε neighborhood of z_0 which is a subset of S.
- A point z_0 is an ***exterior point*** of a set $S \subset \mathbb{C}$ if there is some ε neighborhood of z_0 containing no points of S (i.e., disjoint from S).
- A point z_0 is a ***boundary point*** of set $S \subset \mathbb{C}$ if it is neither an interior point nor an exterior point of S.
- A point z_0 is an ***accumulation point*** of set S ⊂ C if *each* deleted neighborhood of z_0 contains at least one point of S.

> **Question1**
>
> **Find the set of interior points, boundary points, accumulation points, and isolated points for:**



Answer1

Sets of Points in the Complex Plane

- A set $\Omega \subset \mathbb{C}$ is called ***open*** if for every $z \in \Omega$ there exists an $\varepsilon > 0$ such that $B_\varepsilon(z) = \{w \in \mathbb{C} \mid |w - z| < \varepsilon\} \subset \Omega$. A set is called ***closed*** if its complement is open.
- A set $\Omega \subset \mathbb{C}$ is called ***bounded*** if $\Omega \subset B_R(0)$ for some $R > 0$.
- A set $K \subset \mathbb{C}$ is called ***compact*** if every sequence in K has a subsequence that converges in K . A set $K \subset \mathbb{C}$ is compact if and only if it is *closed* and *bounded*.
- An open (closed) set $\Omega \subset \mathbb{C}$ is called ***disconnected*** if there exist two open (closed) sets Ω_1, $\Omega_2 \subset \mathbb{C}$ such that $\Omega_1 \cap \Omega_2 = \emptyset$ and $\Omega = \Omega_1 \cup \Omega_2$.
- If Ω is not disconnected, Ω is called ***connected***. A set $\Omega \subset \mathbb{C}$ is connected if and only if for any two points in Ω there exists a curve joining them.
- An *open* and *connected* set is called a ***domain***, or ***region***.
- Define the ***diameter*** of a set $\Omega \subset \mathbb{C}$ by

$$\operatorname{diam}(\Omega) := \sup_{z,w \in \Omega} |z - w|$$

Question2

> **Give a set which is open and closed.**
>
> **Give a set which is closed and unbounded.**

> **Answer2**

> **Question3**
>
> **Is the set $U = \{Z \in \mathbb{C} : 2 < |z| \leq 3\}$ open or closed?**

> **Answer3**

Functions in the Complex Plane

$$f : \mathbb{C} \to \mathbb{C}, \quad f(x + iy) = u(x, y) + iv(x, y)$$

- Differentiable / holomorphic
- Analytic

Complex Differentiability

Definition of Holomorphic

We say that a function $f : \mathbb{C} \to \mathbb{C}$ is ***complex differentiable***, or ***holomorphic***, at $z \in \mathbb{C}$ if

$$f'(z) := \lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f(z + h) - f(z)}{h}$$

A function is holomorphic on an open set $\Omega \subset \mathbb{C}$ if it is holomorphic at every $z \in \Omega$. A function that is holomorphic on $\mathbb{C}$ is called ***entire***.

Decide Holomorphic

The Cauchy-Riemann Differential Equations

If f is ***holomorphic***, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

And suppose that *the partial derivatives of u and v exist*, are *continuous* and *satisfy the Cauchy-Riemann equations*. Then *f* is ***holomorphic***.

A Second look

Define two operators:

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)$$

If f is ***holomorphic***, then

$$f'(z) = \frac{\partial f}{\partial z} = \frac{\partial u}{\partial z} + i\frac{\partial v}{\partial z} = 2\frac{\partial u}{\partial z} \quad \text{and} \quad \frac{\partial f}{\partial \bar{z}} = 0$$

> **Question4**
>
> **Decide whether the complex variable function f is differentiable:**
>
> $$f(x, y) = \frac{x - 1 - iy}{(x-1)^2 + y^2}$$

> **Answer4**
>
> **Hint: In addition to the obvious way, can you prove by the substitutions** $x = \frac{z+\bar{z}}{2}$ **and** $y = \frac{z-\bar{z}}{2i}$ **?**

A Third Look

Define $u(x, y)$ to be a ***harmonic function*** if:

$$\Delta u = u_{xx} + u_{yy} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Define $u(x, y)$ and $v(x, y)$ to be a ***harmonic conjugate*** if:

$$f = u + iv$$

is differentiable.

If f is ***holomorphic***, then u, v are ***harmonic***.

A Special Case-Power Series

The power series

$$f(z)=\sum_{n=0}^{\infty} a_n z^n$$

defines a holomorphic function in its disc of convergence. The (complex) derivative of f is also a power series having the same radius of convergence as f, that is,

$$f'(z)=\sum_{n=1}^{\infty} n a_n z^{n-1}$$

A ***power series*** is ***infinitely complex differentiable*** in its disc of convergence, and the higher derivatives are also power series obtained by termwise differentiation.

Analytic Functions

Definition of Analytic

A function f defined on an open set $\Omega \subset \mathbb{C}$ is said to be analytic (or have a power series expansion) at a point $z_0 \in \Omega$ if there exists a power series *centered at* z_0, with *positive* radius of convergence, such that

$$f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

for all z in a neighborhood of z_0. If f has a power series expansion at every point in Ω, we say that f is analytic on Ω.

- **Useful Remark:** The exponential, sine and cosine functions are (by our definition) analytic at 0 and have an infinite radius of convergence. They are automatically defined for all complex numbers.

Analytic and Holomorephic

A holomorphic function is automatically analytic.

Complex Integrals

Definition

- A ***parametrized curve*** is a set $\mathcal{C} \subset \mathbb{C}$ such that there exists a parametrization

  $$\gamma : I \to \mathcal{C}$$

  for some interval I → C, where γ is locally injective. We will say that C is smooth if there exists a parametrization γ that is differentiable with $\gamma'(t) \neq 0$ for all $t \in I$.

  Positively and negatively ***oriented***: parametrized in a counter-clockwise and clockwise fashion, respectively.

- Let $\Omega \subset \mathbb{C}$ be an open set, f holomorphic on Ω and $\mathcal{C}^* \subset \Omega$ an oriented smooth curve. We then define the ***integral*** of f along $\mathcal{C}^*$ by

  $$\int_{\mathcal{C}^*} f(z)dz := \int_I f(\gamma(t)) \cdot \gamma'(t)dt = \int_I [u(\gamma(t)) + iv(\gamma(t))] \cdot \gamma'(t)dt$$

> Though the most basic definition should be in the below form, sometimes useful for calculation.
>
> $$\int_C f(z)dz = \int_C (u + \mathrm{i}v)(dx + \mathrm{i}dy) = \int_C (udx - vdy) + \mathrm{i}\int_C (vdx + udy)$$

- Define the ***curve length*** as

  $$\ell(\mathcal{C}) := \left| \int_{\mathcal{C}} dz \right|$$

Basic Property

$$\int_{-\mathcal{C}^*} f(z)dz = -\int_{\mathcal{C}^*} f(z)dz$$

$$\left| \int_{\mathcal{C}^*} f(z)dz \right| \leq \ell(\mathcal{C}) \cdot \sup_{z \in \mathcal{C}} |f(z)|$$

> **Question5**
>
> **Evaluate the integral, where C the line segment with initial point −1 and final point i; or the arc of the unit circle $Imz \geq 0$ with initial point −1 and final point i.**
>
> $$\int_C |z|^2 dz$$

> **Answer5**

Independence of Path

If a continuous function f has a ***primitive*** F in Ω, and $\mathcal{C}^*$ is a curve in Ω that begins at w_1 and ends at w_2, then

$$\int_{\mathcal{C}^*} f(z) dz = F(w_2) - F(w_1)$$

This is equivalent to

$$\oint_{\mathcal{C}} f(z) dz = 0$$

> A reminder: Does a horlomorephic function f always have a primitive? Recall $f(z) = 1/z$.
>
> Of course not. A horlomorephic function f defined on an open subset of $\mathbb{C}$ which is also simply connected will have a primitive F.

Judgement - Basic

- Goursat's Theorem:

  Let $\Omega \subset \mathbb{C}$ be open and f ***holomorphic*** on Ω. Let $T \subset \Omega$ be a triangle whose ***interior*** is also contained in Ω. Then

  $$\oint_T f(z) dz = 0$$

- Corollary:

  If f is ***holomorphic*** in an open set Ω that contains a rectangle R and its ***interior***, then

  $$\oint_R f(z) dz = 0$$

- Theorem:

  A ***holomorphic*** function in an ***open disc*** has a ***primitive*** in that disc.

- Cauchy's Theorem:

  If f is ***holomorphic*** in a ***disc***, then for any closed curve $\mathcal{C}$ in that disc.

  $$\oint_{\mathcal{C}} f(z) dz = 0$$

- **Cauchy's Integral Theorem*:**

  Let U be an open subset of $\mathbb{C}$ which is ***simply connected***, let $f : U \to \mathbb{C}$ be a ***holomorphic*** function, for any closed curve $\mathcal{C}$ in U

  $$\oint_{\mathcal{C}} f(z) dz = 0$$

- Corollary:

  Suppose f is ***holomorphic*** in an open set $\Omega \subset \mathbb{C}$ containing a circle $\mathcal{C}_0$ and its ***interior***. Then

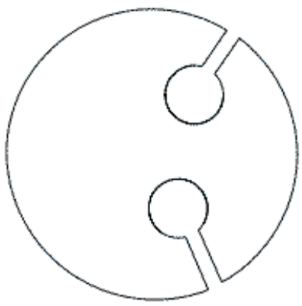
  $$\oint_{C_0} f(z) dz = 0$$

> All of the above theorems has one same key point: the existence of primitive in some region, requires there's no "holes" in the region.

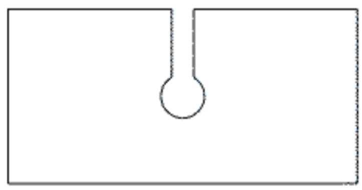
> **Question6**
>
> **C is the unit circle centered at the origin. Explain, relating to the above theorems, why the below integral does not vanishes to 0. You can draw.**
>
> $$\int_C \frac{1}{z} dz$$
>
> ---
>
> **Answer6**

Judgement - Toy Contours

- Cauchy's theorem can be applied to various contours. Below are some toy contours.

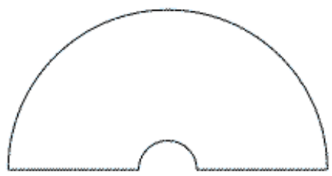


The multiple keyhole

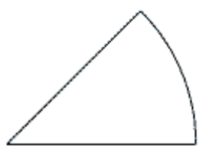


Rectangular keyhole

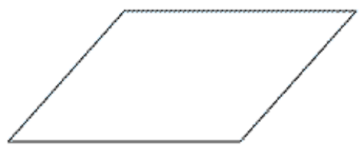
Semicircle



Indented semicircle



Sector



Parallelogram

Simply means: If f is ***holomorphic*** in a ***contour***, then for any closed curve $\mathcal{C}$ in that contour (usually we simply choose the boundary of the contour):

$$\oint_c f(z)dz = 0$$

> This is acually still a special case for the general Cauchy's Integral Theorem*.
>
> A very useful technieque to eveluate integrations and so on.

Jordan's Lemma

Assume that for some $R_0 > 0$ the function $g : \mathbb{C} \backslash \overline{B_{R_0}(0)} \rightarrow \mathbb{C}$ isholomorphic. Let

$$f(z) = e^{iaz} g(z), \quad \text{for some } a > 0$$

Let

$$C_R = \left\{ z \in \mathbb{C} : z = R \cdot e^{i\theta}, 0 \leq \theta \leq \pi \right\}$$

be a semi-circle segment in the upper half-plane and assume that

$$\sup_{0 \leq \theta \leq \pi} \left| g\left(Re^{i\theta}\right) \right| \stackrel{R \rightarrow \infty}{\longrightarrow} 0$$

Then

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z)dz = 0$$

Cauchy Integral Formulas

Suppose f is a holomorphic function in an open set $\Omega \subset \mathbb{C}$. If D is an open disc whose boundary is contained in Ω, then

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} d\zeta \quad \text{for all } z \in D$$

where $C = \partial D$ is the (***positively oriented***) boundary circle of D.

> Tricky question: does it matters whether z is in the disk or not? Draw graphs and analysis.

- The values of a holomorphic function within a disc are fixed by the values of the function on the boundary

> Tricky reminder: does this means all the values of f(z) in a chosen disk are the same?

- Cauchy's integral formula is also valid for all of our toy contours

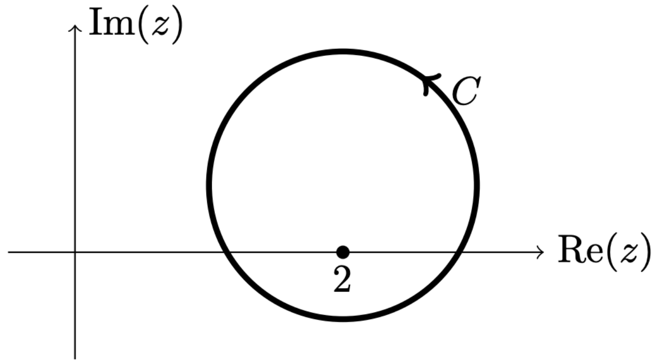
Corollary:

If f is a holomorphic function in an open set $\Omega \subset \mathbb{C}$, then f has infinitely many complex derivatives in Ω. Moreover, if D is an open disc whose boundary is contained in Ω,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \quad \text{for all } z \in D$$

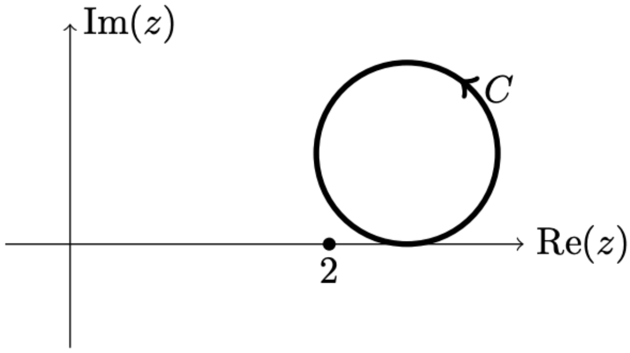
where $C = \partial D$ is the (***positively oriented***) boundary circle of D.

> **Question7**
>
> **Compute $\int_c \frac{e^{z^2}}{z-2} dz$, where C is the curve shown below**



> **Answer7**

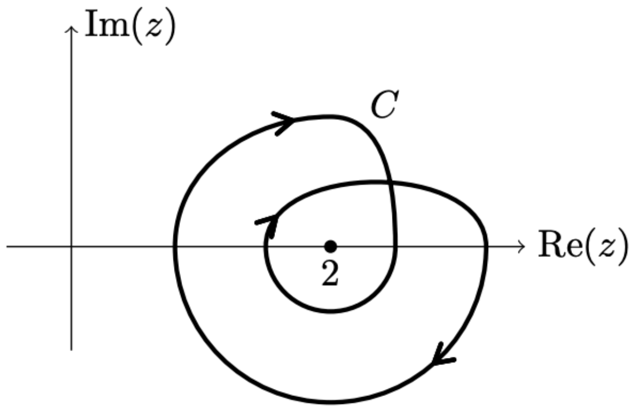
> **Question8**
>
> **Compute $\int_c \frac{e^{z^2}}{z-2} dz$, where C is the curve shown below**



Answer8

Question9

Compute $\int_C \frac{e^{z^2}}{z-2} dz$ **, where C is the curve shown below**



Answer9

Question10

Compute $\int_C \frac{1}{(z-2)(z-5)} dz$ **, where C is the circle with radius 3 and centered at the origin.**

Answer10

Holomorphic Functions are Analytic

Suppose f is a holomorphic function in an open set Ω. If D is an open disc centered at z_0 and whose closure is contained in Ω, then f has a power series expansion at z_0

$$f(z)=\sum_{n=0}^{\infty} a_n\left(z-z_0\right)^n$$

for all $z \in D$ and the coefficients are given by

$$a_n=\frac{f^{(n)}\left(z_0\right)}{n!}, \quad n \in \mathbb{N}$$