Solve Linear Systems

1. Linear Algebra as a Tool

(1) Eigenvalue Problem.

V: A real or complex vector space

L: A linear transformation $V \to V$

$\lambda \in \mathbb{F}$

key equation: $Lx = \lambda \cdot x$ $\quad (*)$

① Eigenvalue: $\lambda \in \mathbb{F}$, where exits $x \in V$ s.t. $(*)$ holds

② Eigenvector: $x \in V$, where $(*)$ holds for certain λ. $x \neq \vec{0}$

③ Eigenspace: $V_\lambda = \{x \in V : Lx = \lambda x\}$ for an eigenvalue λ.

(2) Solve EP for matrices

$A \in \mathbb{C}^{n \times n}$, $\quad Ax = \lambda x \iff (A - \lambda \mathbb{1})x = 0$.

① Find λ

Solutions x exist iff $\det(A - \lambda \mathbb{1}) = 0$

$p(\lambda) = \det(A - \lambda \mathbb{1}) = 0$ gives λ.

characteristic polynomial, of degree n, has at most n distinct roots.

② Find V_λ, and basis of V_λ to be eigenvectors for each λ.

(3) ① **Algebraic Multiplicity** for λ: Repeating times in $p(\lambda)$

$\geq$

② **Geometric Multiplicity** for λ: $\dim(V_\lambda)$.

(4) Diagonalizable Matrices

① Question: A has n distinct eigenvalues λ_i ?

No. A has n distinct eigenvectors $\vec{v}_i$. One λ_k can have ≥ 1 $\vec{v}_i$

$$U = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}, \quad D = U^{-1} A U = \begin{bmatrix} \lambda_1 & & \cdots & 0 \\ & \lambda_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \lambda_n \end{bmatrix}$$

All $\vec{v}_i$ distinct, λ_i can differ.

Question: If certain λ_k is complex, will λ_k still have eigenvectors ?

Yes. Conisder everything in $\mathbb{C}$. No matter whether $A \in \mathbb{R}^{n\times n}$

② $A^k = U D^k U^{-1}$, $\quad D^k = \begin{bmatrix} \lambda_1^k & & \cdots & 0 \\ & \lambda_2^k & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \lambda_n^k \end{bmatrix}$,

$$e^{At} = \mathbb{1} + \sum_{k=1}^{\infty} \frac{A^k t^k}{k!}$$

$$e^{Dt} = \mathbb{1} + \sum_{k=1}^{\infty} \frac{D^k t^k}{k!} = \begin{bmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$$

Quicker ways to get e^{At} ?

③ Functional Calculus

$f(x) = \sum_{j=0}^{\infty} c_j x^j$, having infinite radius of convergence.

$$f(A) = \sum_{j=0}^{\infty} c_j A^j = \sum_{j=0}^{\infty} c_j (UD^jU^{-1}) = U\left(\sum_{j=0}^{\infty} c_j D^j\right)U^{-1}$$

$$= U\begin{bmatrix} \sum_{j=0}^{\infty} c_j \lambda_1^j & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sum_{j=0}^{\infty} c_j \lambda_n^j \end{bmatrix}U^{-1} = U\begin{bmatrix} f(\lambda_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f(\lambda_n) \end{bmatrix}U^{-1}$$

④ Given by ②.

Important properties:

$$e^{At} = U\begin{bmatrix} e^{\lambda_1 t} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & e^{\lambda_n t} \end{bmatrix}U^{-1} = Ue^{Dt}U^{-1}$$

⑤ The spectral theorem:

Every self-adjoint matrix A is diagonalizable.

* Self-adjoint: $A = A^* = \bar{A}^T$

Definition of adjoint.

$$\langle x, Ay \rangle = \langle A^* x, y \rangle$$

if A is diagonalizable, $e^{At} = Ue^{Dt}U^{-1}$ is easy to calculate e^{At}.

what if A is not diagonalizable? How can we get e^{At}?

e.g. not enough independent eigenvectors?

(5) Non-diagonalizable Matrices

① Summary : 1° Find generalized eigenvectors $v_1, \cdots, v_n$ to form U.

$\downarrow$

2° Define Jordan matrices as $J = U^{-1}AU$,

which can actually be written out after find all λ_i and v_i.

$\downarrow$

3° $e^{At} = Ue^{Jt}U^{-1}$

$e^{Jt} = e^{Dt}e^{Nt}$ $\quad$ ($J = D+N$, where N^k for some k)

($\because$ $ND = DN$.)

② Find generalized eigenvectors by "Bottom-up".

For each λ such that $\dim V_\lambda < a_\lambda$

$$\begin{cases} E_1 = V_\lambda = \ker(A - \lambda\mathbb{1}) \\ E_k = \ker(A - \lambda\mathbb{1})^k \end{cases}$$

Choose $v_i^{(1)} \in E_1$, solve $\begin{cases} (A-\lambda\mathbb{1})v^{(2)} = v^{(1)} \\ \vdots \\ (A-\lambda\mathbb{1})v^{(k+1)} = v^{(k)} \end{cases}$ until you find

as much vectors as a_λ. make sure $v^{(k)} \in E_k \setminus E_{k-1}$

If certain $v_i^{(1)}$ can not find more solutions, choose another $v_j^{(1)} \in E_1$ and again start from the beginning.

③ Find generalized eigenvectors by "Top-down"

For each λ such that $\dim V_\lambda < a_\lambda$, set $m = a_\lambda - \dim V_\lambda + 1$

then solve $\begin{cases} (A-\lambda\mathbb{1})^m v = 0 \\ (A-\lambda\mathbb{1})^{m-1} v \neq 0 \end{cases}$ as $v^{(m)}$.

Get $v^{(m-1)} = (A-\lambda\mathbb{1})v^{(m)}$, $\cdots$, $v^{(1)} = (A-\lambda\mathbb{1})v^{(2)}$.

* Notice $v^{(1)} \in E_1$.

④ Write out Jordan Matrices J.

$$J = \begin{bmatrix} J_{k_1}(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & J_{k_m}(\lambda_m) \end{bmatrix} \qquad J_{k_i} = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$$

Question: Sould $\lambda_1, \dots, \lambda_m \in \mathbb{C}$ all be distinct?

No.

Question: Is J unique?

No. U decides.

Exercise:

$A \in \mathbb{R}^{12 \times 12}$, $\quad \lambda_1: v_1 \ v_2^{(1)} \ v_2^{(2)}$

$\lambda_2: w_1^{(1)} \ w_1^{(2)} \ w_2^{(1)} \ w_2^{(2)} \ w_2^{(3)}$

$\lambda_3: z_1 \ z_2 \ z_3 \ z_4$

Let U be $\begin{bmatrix} | & | & | & | & | & | & | & | & | & | & | & | \\ v_1 & v_2^{(1)} & v_2^{(2)} & w_1^{(1)} & w_1^{(2)} & w_2^{(1)} & w_2^{(2)} & w_2^{(3)} & z_1 & z_2 & z_3 & z_4 \\ | & | & | & | & | & | & | & | & | & | & | & | \end{bmatrix}$

What is the Jordan matrix of A? How many Jordan blocks are there?

Answer:

$$\begin{bmatrix} \lambda_1 & 0 & 0 & & & & & & & 0 & & \\ & \lambda_1 & 1 & & & & & & & & & \\ & & \lambda_1 & 0 & & & & & & & & \\ & & & \lambda_2 & 1 & & & & & & & \\ & & & & \lambda_2 & 0 & & & & & & \\ & & & & & \lambda_2 & 1 & & & & & \\ & & & & & & \lambda_2 & 1 & & & & \\ & & & & & & & \lambda_2 & 0 & & & \\ & & & & & & & & \lambda_3 & 0 & & \\ & & 0 & & & & & & & \lambda_3 & 0 & \\ & & & & & & & & & & \lambda_3 & 0 \\ & & & & & & & & & & & \lambda_3 \end{bmatrix}$$

1 2 3 4 5 6 7 8

⑤ $e^N = \sum_{i=0}^{\infty} \frac{1}{i!} N^i$

$e^{Nt} = \sum_{i=0}^{\infty} \frac{1}{i!} N^i t^i$

2. Homogeneous Solution

$\dot{x} = Ax \qquad x(t_0) = x_0$

(1) Originally :

$t_0 = 0, \quad x(t) = e^{At} x(0) = \left[\mathbb{1} + \sum_{k=1}^{\infty} \frac{A^k t^k}{k!} \right] x(0)$

$t_0 \neq 0, \quad x(t) = e^{A(t-t_0)} x_0 \quad$ (Why ?)

$\forall\ t' = t - t_0$

$\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases}$

$x = e^{At'} x_0 = e^{A(t-t_0)} x_0$

where we write $e^{At} = X(t)$ is the fundamental matrix.

$\downarrow$ how to calculate e^{At} conveniently ?

(2) Let $U = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$, where $\vec{v}_i$ are (generalized) eigenvectors of A.

$\forall\ U\vec{e}_i = \vec{v}_i, \quad U^{-1}\vec{v}_i = \vec{e}_i.$

$J = U^{-1} A U, \quad A = U J U^{-1}$

key result : $X(t) = e^{At} = U e^{Jt} U^{-1}$

$\downarrow$ Notice $e^{At}\vec{v}_i = U e^{Jt}(U^{-1}\vec{v}_i) = U e^{Jt}(\vec{e}_i)$,

Ue^{Jt} would still be a fundamental system.

but may not be a foundamental matrix.

(3) Write $J = D + N$, where D only has diagnals.

key result : $X^{*}(t) = U e^{Jt} = U e^{Dt} e^{Nt}$

Then $x(t) = X^{*}(t)\, \vec{x}_i$, need to solve $\vec{x}_i$ using $x(t_0) = x_0$

Exercise:

Find a fundamental system of the equation:

$$\dot{x} = Ax. \qquad A = \begin{bmatrix} 2 & -1 & -1 \\ -2 & 1 & 3 \\ 0 & -1 & 1 \end{bmatrix}.$$

Solution:

$$\det(A - \lambda \mathbb{1}) = (2-\lambda)\left((\lambda-1)^2+1\right)$$

$$\lambda_1 = 2, \quad \lambda_2 = 1+i, \quad \lambda_3 = 1-i$$

$A \in \mathbb{R}^{n\times n}$, complex λ appear in conjugate pairs.

① $(A - 2\mathbb{1})\vec{v}_1 = 0$

$$\begin{bmatrix} 0 & -1 & -1 \\ -2 & -1 & 3 \\ 0 & -1 & -1 \end{bmatrix}\vec{v} = 0 \quad \text{gives one } \vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

② $(A - (1+i)\mathbb{1})\vec{v}_2 = 0$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -i \\ 1 \end{bmatrix}$$

③ $(A - (1-i)\mathbb{1})\vec{v}_3 = 0$

$$\vec{v}_3 = \begin{bmatrix} 1 \\ i \\ 1 \end{bmatrix}$$

$$X(t) = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{(1+i)t} & 0 \\ 0 & 0 & e^{(1-i)t} \end{bmatrix} = \begin{bmatrix} | & | & | \\ \vec{v}_1 e^{2t} & \vec{v}_2 e^{(1+i)t} & \vec{v}_3 e^{(1-i)t} \\ | & | & | \end{bmatrix}$$

$[(\cdots) + (\cdots)]/2$, $[(\cdots) - (\cdots)]/2$ gives

$$e^x \begin{bmatrix} \cos x \\ \sin x \\ \cos x \end{bmatrix}, \qquad e^x \begin{bmatrix} \sin x \\ -\cos x \\ \sin x \end{bmatrix}. \qquad X_2(t) = \begin{bmatrix} 2e^{2x} & e^x\cos x & e^x \sin x \\ -e^{2x} & e^x \sin x & -e^x\cos x \\ e^{2x} & e^x \cos x & e^x \sin x \end{bmatrix}.$$

3. Particular Solution

$$\dot{x} = Ax + b(t) \qquad x(t_0) = 0$$

(1) $$e^{-At}\frac{dx}{dt} = Ae^{-At}x + e^{-At}b(t)$$

$$\Rightarrow \frac{d}{dt}(e^{-At}x) = e^{-At}b(t)$$

$$\Rightarrow x_{part} = e^{At}\int_{t_0}^{t} e^{-As}b(s)\,ds$$

(2) Variation of Parameters (Can also be applied if $A = A(t)$)

① $$x_{part}(t) = C_1(t)x^{(1)}(t) + \cdots + C_n(t)x^{(n)}(t)$$
$$= X(t)\,C(t), \quad \text{where } C(t) = \begin{pmatrix} C_1(t) \\ \vdots \\ C_n(t) \end{pmatrix}$$

② Solve $X(t)\,C'(t) = b(t)$ to get $C(t)$,

by using Cramer's rule,

$$C_k'(t) = \frac{\det X^{(k)}(t)}{\det X(t)} = \frac{W^{(k)}(t)}{W(t)}.$$

where

- $X^{(k)}$ is the fundamental matrix where the kth column has been replaced with b,
- $W(t) = \det X(t)$ is the Wronskian,
- $W^{(k)}(t) = \det X^{(k)}(t)$.

so $$C_k(t) = \int \frac{W^{(k)}(t)}{W(t)}\,dt.$$