VV286 Recitation Class Note2

Higher-order ODE & Systems of ODEs

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1 Overview

1. System of Equations
2. The Eigenvalue Problem
3. The Spectral Theorem for Self-Adjoint Matrices
4. Non-Diagonalizable Matrices
5. Solutions to Inhomogeneous, Linear Systems
6. Linear Second-Order Equations and Vibrations

2 System of Equations

2.1 Explicit Systems of n first-order Differential Equations

Explicit systems of n first-order differential equations have the form

$$\dot{x}(t) = F(x, t)$$

where

$$x : \mathbb{R} \to \mathbb{R}^n, \quad F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$$

2.2 Transform Higher-order ODE to System of Equations

Given an explicit ODE of order n,

$$x^{(n)}(t) = f\left(x, x', x'', \ldots, x^{(n-1)}, t\right)$$

introduce "new variables" by setting

$$x_1 := x, \quad x_2 := x', \quad x_3 := x'', \quad \ldots, \quad x_n := x^{(n-1)}$$

We then rewrite a n-order ODE as a system of equations

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ \vdots \\ x_n'(t) \end{pmatrix} = \begin{pmatrix} x_2(t) \\ x_3(t) \\ x_4(t) \\ \vdots \\ f\left(x_1, x_2, \ldots, x_n, t\right) \end{pmatrix}$$

Therefore, our general is to solve a IVP involving explicit systems of n first-order differential equations.

$$\frac{dx}{dt} = F(x, t), \quad x\left(t_0\right) = x_0 \in \mathbb{R}^n$$

Next, how?

2.3 Picard Iteration

This is a method for you to gain an approximate solution.

Guess a function $x^{(0)}(t)$, e.g., $x^{(0)}(t) = x_0$ (constant). Then set

$$x^{(k+1)}(t) := x_0 + \int_{t_0}^{t} F\left(x^{(k)}(s), s\right) ds, \quad k \in \mathbb{N}$$

This yields a sequence of functions $(x^{(k)}(t))$.

Under suitable conditions on the function F, $(x^{(k)}(t))$ converges to a (unique) function x(t). (How do you understand?)

So by iterating enough times, you can gain an approximated solution.

Exercise 1

Using Picard iteration to solve the below IVP:

$$\frac{dx}{dt} = xt, \quad x(0) = 1$$

2.4 Analyzing the Solution of the General IVP

2.4.1 Existence and Uniqueness

Theorem of Picard-Lindelö̈f

Let $x_0 \in \Omega$, where $\Omega \subset \mathbb{R}^n$ is open and let $t_0 \in I$, where $I \subset \mathbb{R}$ is an interval. Suppose $F : \Omega \times l \to \mathbb{R}^n$ is a continuous function satisfying a Lipschitz estimate in x: there exists an L > 0 such that for all $x, y \in \Omega$ and all $t \in I$

$$\|F(x,t) - F(y,t)\| \leqslant L\|x - y\|$$

then the initial value problem has a unique solution in some t-interval containing t_0.

2.4.2 Stability

Gronwall's Inequality

Under same conditions, x , y satisfies

$$x'(t) = F(x,t), x\left(t_0\right) = x_0, y'(t) = F(y,t), y\left(t_0\right) = y_0$$

Then

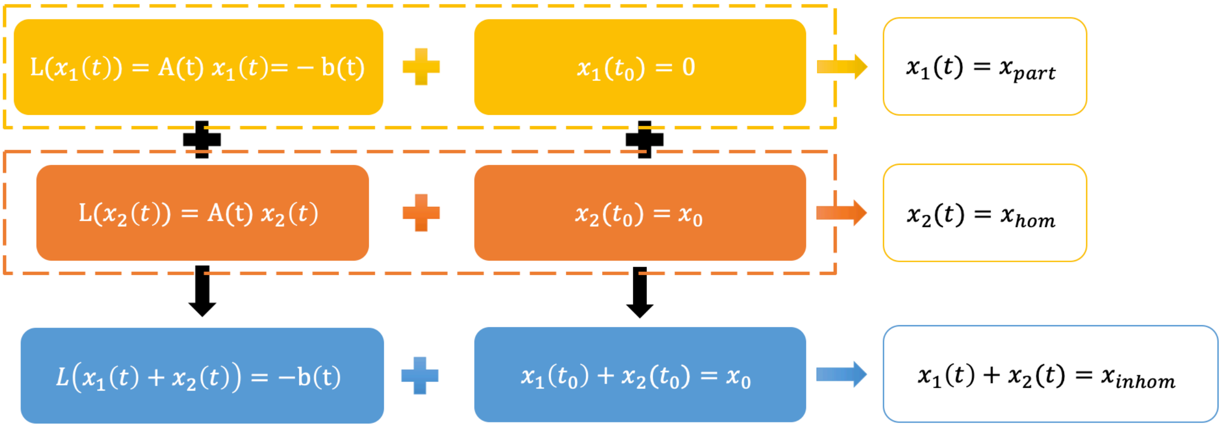
$$\|x(t) - y(t)\| \leqslant e^{L\cdot|t-t_0|}\left\|x_0 - y_0\right\|$$

2.5 Linear System of ODEs

2.5.1 General View of the Solutions

$$\frac{dx}{dt} = F(x,t) = A(t)x + b(t), \quad x(t_0) = x_0 \in \mathbb{R}^n$$

where we mainly focus on $A(t) = A$ is constant.



The graph give you an overview of how we will solve the linear system of ODEs. Next is how can we solve it?

2.5.2 Construction of Solutions for x_{hom}

$$\frac{dx}{dt} = A(t)x, \quad x(t_0) = x_0 \in \mathbb{R}^n$$

If you have n solutions $x^{(1)}(t)$, ..., $x^{(n)}(t)$, where b_k is the base vector, and $x^{(k)}(t)$ solves

$$\frac{dx}{dt} = A(t)x, \quad x(t_0) = b_k \in \mathbb{R}^n$$

writing the **fundamental matrix**

$$X(t) = \begin{pmatrix} | & & | \\ x^{(1)}(t) & \dots & x^{(n)}(t) \\ | & & | \end{pmatrix}$$

then

$$x(t) = X(t)x_0$$

solves the IVP. (why?)

Reminder 1

The solution space of the IVP is a subset of

$$span\{x^{(1)}(t), \dots, x^{(n)}(t)\}$$

2.5.3 One Possible way to solve x_{hom}

Thought 1

If you treat A as a constant in R, what's the solution to the below IVP?

$$\frac{dx}{dt} = Ax, \quad x(t_0) = x_0$$

Define:

$$e^{At} := 1 + \sum_{k=1}^{\infty} \frac{A^k t^k}{k!}$$

Reminder 2

Is the operation for e^{At} well-defined? What is "well-defined" and how do we prove it for e^{At}?

Well-defined

H is weel-defined if $\phi \in V$, then $H\phi \in V$

Proof for e^{At}

Use operator norm with euclidean norm in

$$\|A\| = \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{|Ax|}{|x|}$$

Then

$$\sum_{k=1}^{\infty} \left\| \frac{A^k t^k}{k!} \right\| = \sum_{k=1}^{\infty} \frac{\|A^k\| \cdot |t^k|}{k!} \leqslant \sum_{k=1}^{\infty} \frac{\|A\|^k \cdot |t|^k}{k!} = e^{|t| \cdot \|A\|} - 1 < \infty$$

The series is absolutely convergent, and therefore it converges to some matrix in Mat(n $\times$ n, R).

Also we have $\frac{d}{dt} e^{At} = Ae^{At}$, and when t = 0, e^{At} will be the identity matrix.