VV286 Recitation Class Note1

First Order Differantial Equations

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1 About Me

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2 Overview

1. Separable Equations
2. Linear Equations
3. Transformable Equations
4. General Integral Curves of First Order ODEs
5. Implicit Equations

3 Separable Equations

3.1 Theorem 1.1.3

Let η be an interior point of I_y such that $\underline{g(\eta) \neq 0}$ and let (Hyp) hold. Then there exists a neighborhood of ξ in I_x in which the IVP

$$y' = f(x)g(y), \quad y(\xi) = \eta$$

has a unique solution y(x). It can be obtained from

$$G(y) = \int_\eta^y \frac{ds}{g(s)} = \int_\xi^x f(t)dt = F(x)$$

by solving for y.

Reminder 1

What is the solution if $g(\eta) = 0$ in Theorem 1.1.3?

3.2 Equilibrium, Steady-State, Transient Solutions

1. Equilibrium solution:
$$x_{\text{equi}}(t) = \text{ constant}$$
2. Steady-state solution:
$$x_{\text{ss}} = \lim_{t\to\infty} x(t)$$
3. Transient component:
$$x(t) - x_{ss}$$

Exercise 1

Solve the IVP problem:

$$\frac{dy}{dx} = x^4 y + x^4 y^4, \quad y(0) = 1$$

4 Linear Equations

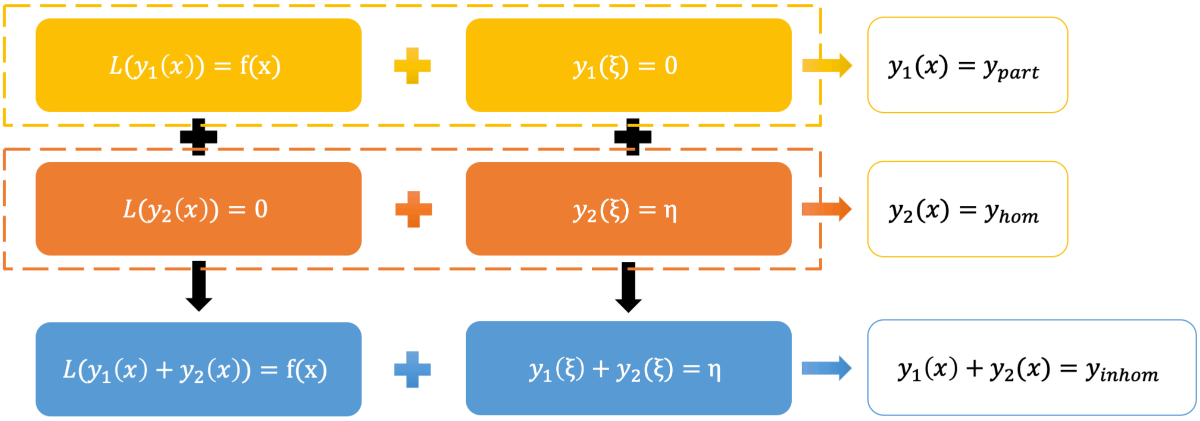
A general linear, first-order ordinary differential equation on an open interval $I \subset \mathbb{R}$ has the form

$$a_1(x)y' + a_0(x)y = f(x), \quad x \in I$$

where we allow a_0, a_1, f to be continuous, real-valued functions on I.

4.1 Analyze the Solutions of General IVP

$$a_1(x)y' + a_0(x)y = f(x), \quad x \in I$$
$$y(\xi) = \eta$$



Thought 1

$L = a_1 \frac{d}{dx} + a_2$ can be viewed as a linear transformation $L : V \to V$. What is the vector space V? Can you define a linear transformation T representing the second equation $y(\xi) = \eta$?

4.2 Solving the General IVP

4.2.1 Solving y_{hom}

Theorem 1.1.3.

4.2.2 Solving y_{part}

$$a_1(x)y' + a_0(x)y = f(x), \quad x \in I$$
$$y(\xi) = 0$$

1. Duhamel's Principle:
Let $I \subset R$ be an open interval, $x_0 \in \bar{I}$, and a_0,a_1, f continuous, real-valued functions on I, where $a_1(x) \neq 0$ for all $x \in \bar{I}$. Let y_ξ solve the initial value problem

$$a_1(x)y_\xi' + a_0(x)y_\xi = 0, \quad y_\xi(\xi) = \frac{1}{a_1(\xi)}$$

for $x \in \bar{I}$. Then

$$y(x) = \int_{x_0}^{x} f(\xi)y_\xi(x)d\xi$$

solves

$$a_1(x)y' + a_0(x)y = f(x), \quad y(x_0) = 0$$

Thought 2

You can follow the proof in slides easily. What about understand Duhamel's Principle physically? Let's draw a graph. (Hint: similar to what we have done in the radioactive decay problem.)

2. Variation of Parameters:
Let $y_{part}(x) = c(x)y_{hom}(x)$, then we can solve c(x) from the below equations(why?), and then find $y_{part}(x)$:

$$a_1(x)c'(x)y_{\text{hom}}(x) = f(x)$$
$$c(\xi) = 0$$

3. Integrating Factors:
First solve u(x) from the below equations:

$$u'(x) = \frac{a_0(x)}{a_1(x)} u(x)$$

$$u(\xi) = 1$$

Second solve h(x) = u(x)y(x) from the below equations(why?):

$$h'(x) = \frac{f(x)u(x)}{a_1(x)}$$

$$h(\xi) = \eta$$

Further y(x) = h(x)/u(x).

Exercise 2

Find a general solution for:

$$\frac{dy}{dx} - \frac{4}{x} y = x^5 e^x$$

5 Transformable Equations

1. $y' = f(ax + by + c); b \neq 0$
Let $u(x) = ax + by + c$,
Then $u'(x) = a + bf(u)$.

2. $y' = f(y/x)$
Let $u(x) = \frac{y(x)}{x}$,
Then $u'(x) = (f(u) - u)\frac{1}{x}$.

3. $y' = f\left(\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\right)$
Let $u(x) = a_1x + b_1y(x) + c_1, v(x) = a_2x + b_2y(x) + c_2$,
Then $x = \frac{b_2(u-c_1)-b_1(v-c_2)}{a_1b_2-a_2b_1}$,
And $\frac{du}{dv}$ can be simplified to a form $\frac{du}{dv} = h(u/v)$(how?),
Hence you can solve $u(x) = c(v)$ following 5.2,
And then obtain the relationship between x and y.

Exercise3

Find a general solution for:

$$y' = \frac{x - y}{x + y}$$

4. $y' + gy + hy^\alpha = 0, \alpha \neq 1$ **(Bernoulli's equation)**
Let $u(x) = y^{1-\alpha}$,
Multiply both side with u(x),
then $u'(x) + (1-\alpha)g(x)u(x) + (1-\alpha)h(x) = 0$.

5. $y' + gy + hy^2 = k$ **(Ricatti's equation)**
First you guess/know a solution $\phi(x)$,
Let $u(x) = y(x) - \phi(x)$,
From

$$\begin{cases} y' + gy + hy^2 = k \\ \phi' + g\phi + h\phi^2 = k \end{cases}$$

,
You obtain $u' + gu + h\left(y^2 - \phi^2\right) = 0$, which gives $u' + (g + 2\phi h)u + hu^2 = 0$.
Solve this Bernoulli's equation where $\alpha = 2$.

Exercise 4

Solve the IVP:

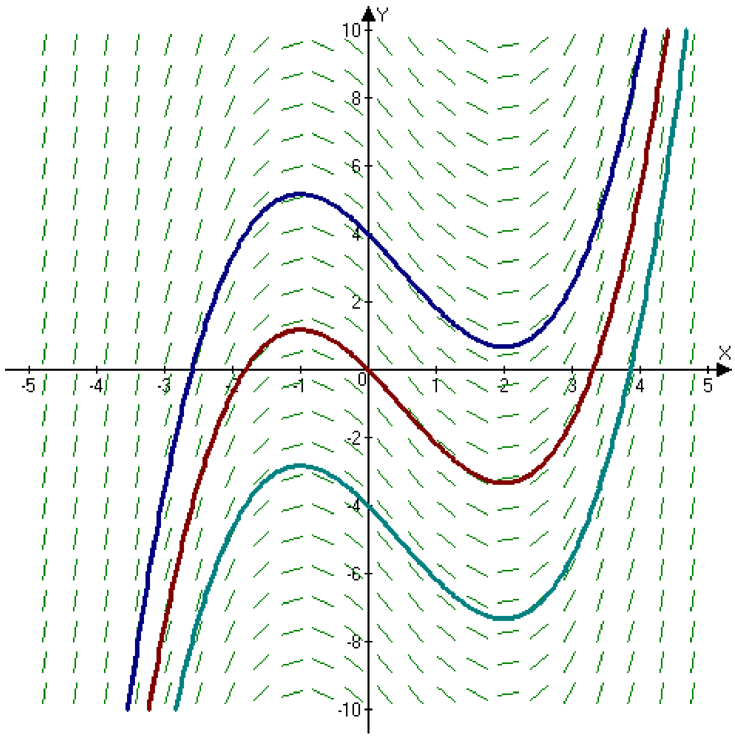
$$\begin{cases} \frac{dy}{dx} = \frac{2\cos^2(x) - \sin^2(x) + y^2}{2\cos(x)} \\ y(0) = -1 \end{cases}$$

6 General Integral Curves of First Order ODEs

6.1 Overview

$$h(x, y)y' + g(x, y) = 0$$
$$y' = -\frac{g(x, y)}{h(x, y)}, \quad if\, h \neq 0$$

1. What is a vector field F(x,y)? What is an integral curve for a vector field F(x,y)?
2. How is an integral curve related to an first-order ODE? (Reminder: what does y' mean graphically? What are the vectors (-h(x,y),g(x,y)) and (g(x,y),h(x,y))?)
3. How to solve an integral curve for a vector field F(x,y)? (Reminder: $F^{\perp}(x, y)$; $U^{\perp}(x, y) = C_0$)
4. What if initially $F^{\perp}(x, y)$ do not have $U^{\perp}(x, y)$?



6.2 Solving ODE with an Integral Curve

1. Judge $\begin{pmatrix} g(x,y) \\ h(x,y) \end{pmatrix}$ whether: $\frac{\partial g(x,y)}{\partial y} = \frac{\partial h(x,y)}{\partial x}$
2. If not, $\begin{pmatrix} M(x,y)g(x,y) \\ M(x,y)h(x,y) \end{pmatrix}$, find M so that $\frac{\partial (Mg)}{\partial y} = \frac{\partial (Mh)}{\partial x}$.

   i Most general: $M_y g + M g_y = M_x h + M h_x$

   ii $M(x,y) = M(x)$: $M'(x)h = M(g_y - h_x) \Rightarrow \frac{M'(x)}{M(x)} = \frac{g_y - h_x}{h} = (\ln M(x))'$. There are always two solvable cases.

3. Find $U^\perp$ for $F^\perp = \begin{pmatrix} Mg \\ Mh \end{pmatrix}$.

   i $\frac{\partial U^t}{\partial x} = Mg \Rightarrow U^\perp = \int Mgdx + f(y)$
   $\frac{\partial (\int Mgdx)}{\partial y} + f'(y) = Mh \Rightarrow get\, f(y)$
   $\Rightarrow get\, U^\perp(t)$

   ii $\begin{cases} \frac{\partial U^\perp}{\partial x} = Mg \\ \frac{\partial U^\perp}{\partial y} = Mh \end{cases} \Rightarrow \begin{cases} U^\perp = \int Mgdx + f(y) \\ U^\perp = \int Mhdy + k(x) \end{cases} \Rightarrow U^\perp$

4. The solution is in the form $U^\perp(x,y) = C_0$.

Exercise 5

Find a general solution for:

$$y' = \frac{x-y}{x+y}$$

7 Implicit Equations

7.1 Overview of Slope Parametrization

$$F\left(y, y^{\prime} ; x\right)=0$$

Example: $\left(y^{\prime}\right)^{2}-x y^{\prime}+y=0$

1. Why does the method of integral curves fail for the above equation?
2. What other method can we make use of the information the equation gives us about $y^{\prime}$?

Given $y^{\prime\prime}$ exists and $y^{\prime\prime} \neq 0$, $y^{\prime}$ is monotonic function of x. We can use slope $p=\frac{dy}{dx}$ to parametrize the solution curve.

$x=x(p)$, $y=y(p)$, gives

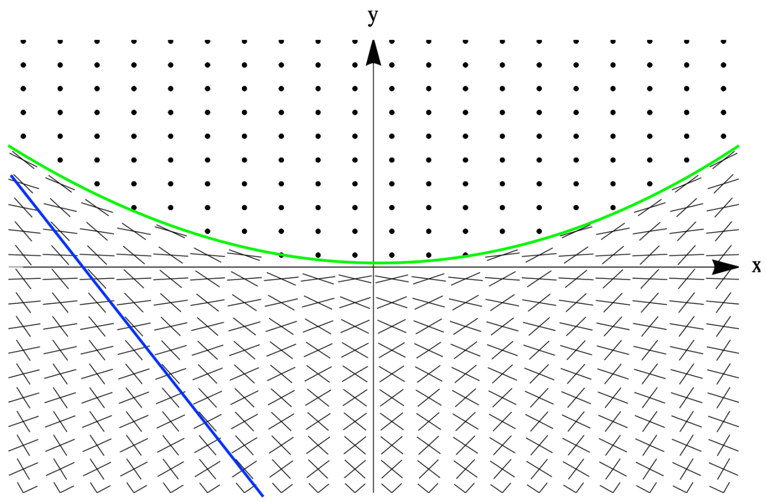
$$\dot{y}(p)=\frac{dy}{dp}=\frac{dy}{dx} \frac{dx}{dp}=p \cdot \dot{x}(p)$$

Further, we can always gain a second relationship between x(p) and y(p) by differentiate F on p: $F\left(y, y^{\prime} ; x\right)=0 \Rightarrow F(y(p), p ; x(p))=0$ and it gives

$$F_{x} \dot{x}+F_{y} \dot{y}+F_{p}=0$$

With the two equations, sometimes you can solve $y=f(x)$ by eliminating p, or just get x(p) and y(p) as solution.

Don't forget the straight line solutions where $y^{\prime\prime}=0$.



7.2 $y = xy' + g(y')$ Clairaut's equation

1. **solve for straight line solutions** $y = cx + g(c)$.

2. **solve for non-trivial solutions** by slope parametrization.

   1. Parametrize $y(p), x(p)$ with slope p, so it gives
   $$\dot{y}(p) = p \cdot \dot{x}(p)$$
   2. Differentiate $y(p) = x(p) \cdot p + g(p)$ to get
   $$\dot{y}(p) = p \cdot \dot{x}(p) + x(p) + \dot{g}(p)$$
   3. Solve and get $x(p) = -g'(p)$ and $y(p) = -pg'(p) + g(p)$.

2. **solve for non-trivial solutions** by finding the envelope.

   1. After finding the straight line solutions $y = cx + g(c)$, parametrize the curve as $\gamma(c, x) = \begin{pmatrix} x \\ cx + g(c) \end{pmatrix}$.
   2. Using $\frac{\partial \gamma_1}{\partial c} \frac{\partial \gamma_2}{\partial x} = \frac{\partial \gamma_1}{\partial x} \frac{\partial \gamma_2}{\partial c}$, we get $x = -g'(c)$, then $y(c) = -cg'(c) + g(c)$.
   3. Get $y = f(x)$ by eliminating c.

Exercise 6

Find a general solution for:
$$x^2 y' + \cos(y + xy') = 0$$

7.3 $y = xf(y') + g(y')$D'Alembert's equation

1. **solve for straight line solutions**. y = cx + d is solution if and only if c = f(c),d = g(c).

2. **solve for non-trivial solutions** by slope parametrization.

1. Parametrize $y(p), x(p)$ with slope p, so it gives

$$\dot{y}(p) = p \cdot \dot{x}(p)$$

2. Differentiate $y(p) = x(p)f(p) + g(p)$ then we get $\dot{y} = \dot{x}f + x\dot{f} + \dot{g}$, which gives

$$x'(p) = \frac{f'(p)x(p) + g'(p)}{p - f(p)}$$

3. Solve x(p), then further get y(p).

Exercise 7

Find a general solution for:

$$y = (y \cdot y' + 2x) \cdot y'$$