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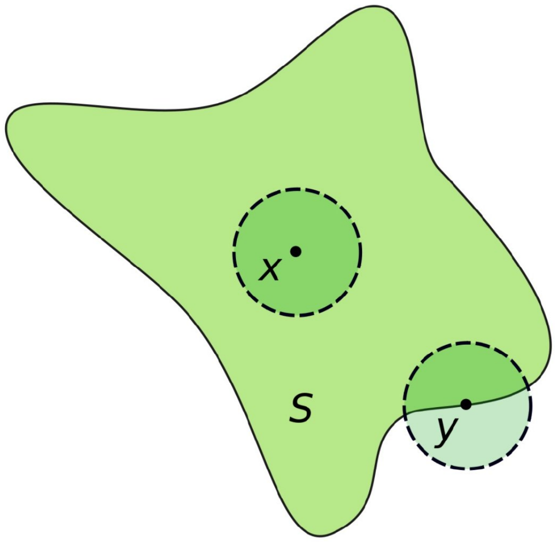
Midterm2 Part1

Conponents in the Complex Plane

Points in the Complex Plane



- For a given $z \in \mathbb{C}$ and $\varepsilon > 0$, the set

  $B_\varepsilon(z) =${$w \in \mathbb{C} \mid |w - z| < \varepsilon$},

  is called an $\varepsilon - \text{neighborhood}$ of z;

  $B_\varepsilon(z) =${$w \in \mathbb{C} \mid 0 < |w - z| < \varepsilon$},

  is called an $\varepsilon - \text{deleted neighborhood}$ of z.
- A point z_0 is an ***interior point*** of set $S \subset \mathbb{C}$ if there is some ε neighborhood of z_0 which is a subset of S.
- A point z_0 is an ***exterior point*** of a set $S \subset \mathbb{C}$ if there is some ε neighborhood of z_0 containing no points of S (i.e., disjoint from S).
- A point z_0 is a ***boundary point*** of set $S \subset \mathbb{C}$ if it is neither an interior point nor an exterior point of S.
- A point z_0 is an ***accumulation point*** of set S ⊂ C if *each* deleted neighborhood of z_0 contains at least one point of S.

Sets of Points in the Complex Plane

- A set $\Omega \subset \mathbb{C}$ is called ***open*** if for every $z \in \Omega$ there exists an $\varepsilon > 0$ such that $B_\varepsilon(z) =${$w \in \mathbb{C} \mid |w - z| < \varepsilon$}$\subset \Omega$. A set is called ***closed*** if its complement is open.
- A set $\Omega \subset \mathbb{C}$ is called ***bounded*** if $\Omega \subset B_R(0)$ for some $R > 0$.
- A set $K \subset \mathbb{C}$ is called ***compact*** if every sequence in K has a subsequence that converges in K . A set $K \subset \mathbb{C}$ is compact if and only if it is *closed* and *bounded*.
- An open (closed) set $\Omega \subset \mathbb{C}$ is called ***disconnected*** if there exist two open (closed) sets Ω_1, $\Omega_2 \subset \mathbb{C}$ such that $\Omega_1 \cap \Omega_2 = \emptyset$ and $\Omega = \Omega_1 \cup \Omega_2$.
- If Ω is not disconnected, Ω is called ***connected***. A set $\Omega \subset \mathbb{C}$ is connected if and only if for any two points in Ω there exists a curve joining them.
- An *open* and *connected* set is called a ***domain***, or ***region***.
- Define the ***diameter*** of a set $\Omega \subset \mathbb{C}$ by

$$\operatorname{diam}(\Omega) := \sup_{z,w\in\Omega} |z-w|$$

Functions in the Complex Plane

$$f : \mathbb{C} \to \mathbb{C}, \quad f(x+iy) = u(x,y) + iv(x,y)$$

Holomorphic Functions

Definition of Holomorphic

We say that a function $f : \mathbb{C} \to \mathbb{C}$ is ***complex differentiable***, or ***holomorphic***, at $z \in \mathbb{C}$ if

$$f'(z) := \lim_{\substack{h\to 0 \\ h\in\mathbb{C}}} \frac{f(z+h)-f(z)}{h}$$

A function is holomorphic on an open set $\Omega \subset \mathbb{C}$ if it is holomorphic at every $z \in \Omega$. A function that is holomorphic on $\mathbb{C}$ is called ***entire***.

The Cauchy-Riemann Differential Equations

1. If f is ***holomorphic***, then the Cauchy-Riemann equations is satisfied:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

2. And suppose that ***the partial derivatives of u and v exist***, are ***continuous*** and ***satisfy the Cauchy-Riemann equations***. Then f is ***holomorphic***.

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3. Define two operators:

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)$$

If f is ***holomorphic***, then

$$f'(z) = \frac{\partial f}{\partial z} = \frac{\partial u}{\partial z} + i\frac{\partial v}{\partial z} = 2\frac{\partial u}{\partial z} \quad \text{and} \quad \frac{\partial f}{\partial \bar{z}} = 0$$

Power Series

The power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

defines a ***holomorphic*** function in its ***disc of convergence***. The (complex) ***derivative*** of f is also a power series ***having the same radius of convergence*** as f, that is,

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

A ***power series*** is ***infinitely complex differentiable*** in its disc of convergence, and the higher derivatives are also power series obtained by termwise differentiation.

Analytic Functions

Definition of Analytic

A function f defined on an open set $\Omega \subset \mathbb{C}$ is said to be analytic (or have a power series expansion) at a point $z_0 \in \Omega$ if there exists a power series *centered at* z_0, with *positive* radius of convergence, such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all z in a neighborhood of z_0. If f has a power series expansion at every point in Ω, we say that f is analytic on Ω.

- **Useful Remark:** The exponential, sine and cosine functions are (by our definition) analytic at 0 and have an infinite radius of convergence. They are automatically defined for all complex numbers.

Holomorphic Functions are Analytic

Suppose f is a holomorphic function in an open set Ω. If D is an open disc centered at z_0 and whose closure is contained in Ω, then f has a power series expansion at z_0

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all $z \in D$ and the coefficients are given by

$$a_n = \frac{f^{(n)}(z_0)}{n!}, \quad n \in \mathbb{N}$$

Complex Integrals

Definition

- A ***parametrized curve*** is a set $\mathcal{C} \subset \mathbb{C}$ such that there exists a parametrization

  $$\gamma : I \to \mathcal{C}$$

  for some interval I → C, where γ is locally injective. We will say that C is smooth if there exists a parametrization γ that is differentiable with $\gamma'(t) \neq 0$ for all $t \in I$.

Understand simply, γ is parametrizing the "position":

$$\gamma(t) = x(t) + iy(t)$$

Positively and negatively ***oriented***: parametrized in a counter-clockwise and clockwise fashion, respectively.

- Let $\Omega \subset \mathbb{C}$ be an open set, f holomorphic on Ω and $\mathcal{C}^* \subset \Omega$ an oriented smooth curve. We then define the ***integral*** of f along $\mathcal{C}^*$ by

$$\int_{\mathcal{C}^*} f(z)dz := \int_I f(\gamma(t)) \cdot \gamma'(t)dt = \int_I [u(\gamma(t)) + iv(\gamma(t))] \cdot \gamma'(t)dt$$

> Though the most basic definition should be in the below form, sometimes useful for calculation.
>
> $$\int_C f(z)dz = \int_C (u(x,y) + \mathrm{i}v(x,y))(dx + \mathrm{i}dy) = \int_C (u(x,y)dx - v(x,y)dy) + \mathrm{i}\int_C (v(x,y)dx + u(x,y)dy)$$

- Define the ***curve length*** as

$$\ell(\mathcal{C}) := \left|\int_{\mathcal{C}} dz\right|$$

Basic Property

- Oriented:

$$\int_{-\mathcal{C}^*} f(z)dz = -\int_{\mathcal{C}^*} f(z)dz$$

- Triangular inequality for integrals:

$$\left|\int_{\mathcal{C}^*} f(z)dz\right| \leq \int_{\mathcal{C}^*} |f(z)|\, dz$$

  * Triangular inequality:

$$|z_1| - |z_2| \leq |z_1 + z_2| \leq |z_1| + |z_2|$$

- Upper bound:

$$\left|\int_{\mathcal{C}^*} f(z)dz\right| \leq \ell(\mathcal{C}) \cdot \sup_{z \in \mathcal{C}} |f(z)|$$

> **Question**
>
> Evaluate the integral along two different paths:
>
> 1. The line segment with initial point −1 and final point i;
> 2. The arc of the unit circle $Im z \geq 0$ with initial point −1 and final point i.

$$\int_C |z|^2 dz$$

> **Answer**

Cauchy's Integral Theorem

Primitive / Independent of Path

If a continuous function f has a ***primitive*** F in Ω, and $\mathcal{C}^*$ is any curve in Ω that begins at w_1 and ends at w_2, then

$$\int_{\mathcal{C}^*} f(z)dz = F(w_2) - F(w_1)$$

This is equivalent to

$$\oint_{\mathcal{C}} f(z)dz = 0$$

> A holomorphic function f defined in a region Ω may not always have a primitive. Recall $f(z) = 1/z$.
>
> One way to judge the existence of primitive F is analyzing the region Ω where the function f is defined.

Cauchy's Integral Theorem

Let U be an open subset of $\mathbb{C}$ which is ***simply connected***, let $f : U \to \mathbb{C}$ be a ***holomorphic*** function, for any closed curve $\mathcal{C}$ in U

$$\oint_{\mathcal{C}} f(z)dz = 0$$

Specific Cases of Cauchy's Integral Theorem

- Goursat's Theorem:

  Let $\Omega \subset \mathbb{C}$ be open and f ***holomorphic*** on Ω. Let $T \subset \Omega$ be a ***triangle*** whose ***interior*** is also contained in Ω. Then

  $$\oint_{T} f(z)dz = 0$$

- Corollary:

  If f is ***holomorphic*** in an open set Ω that contains a ***rectangle*** R and its ***interior***, then

  $$\oint_{R} f(z)dz = 0$$

- Cauchy's Theorem:

  If f is ***holomorphic*** in a ***disc***, then for any closed curve $\mathcal{C}$ in that disc.

$$\oint_{\mathcal{C}} f(z)dz = 0$$

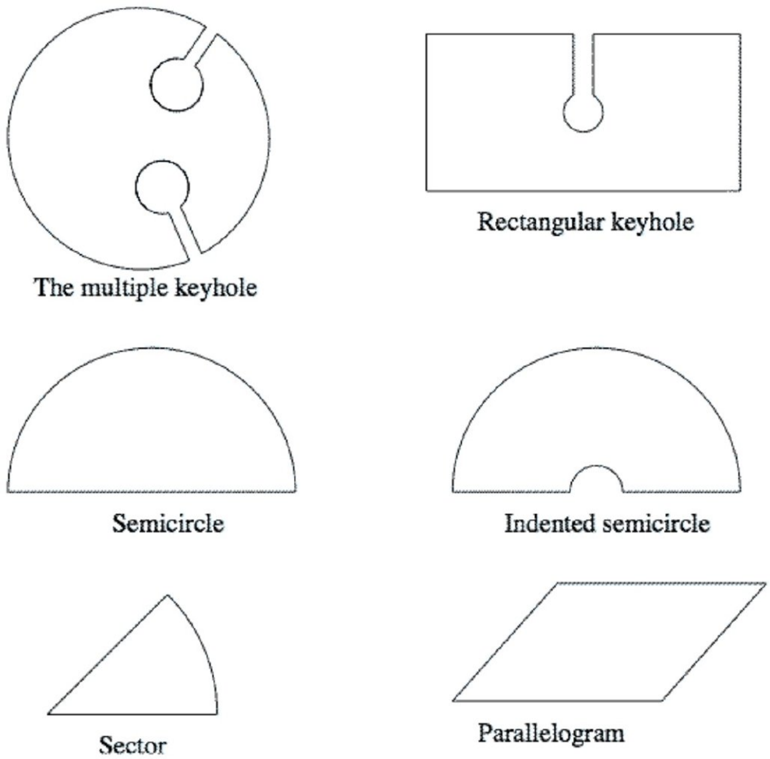
- Corollary:

  Suppose f is ***holomorphic*** in an open set $\Omega \subset \mathbb{C}$ containing a ***circle*** $\mathcal{C}_0$ and its ***interior***. Then

$$\oint_{C_0} f(z)dz = 0$$

- **<u>Toy Contours:</u>**

  Suppose f is ***holomorphic*** in an open set $\Omega \subset \mathbb{C}$ containing a ***toy contour*** and its ***interior***. Then



  Simply means: If f is ***holomorphic*** in a ***contour***, then for any closed curve $\mathcal{C}$ in that contour (usually we simply choose the boundary of the contour):

$$\oint_{c} f(z)dz = 0$$

> **Comment on a special case:**

All z^n has a primitive except for the case where $n = -1$.

$$\oint_{S^1} \frac{dz}{z} = \int_0^{2\pi} \frac{ie^{it}}{e^{it}} dt = 2\pi i \neq 0$$

$$
\begin{aligned}
\oint_{S^1} \frac{dz}{z^n} &= \int_0^{2\pi} \frac{ie^{it}}{e^{nit}} dt = i \int_0^{2\pi} e^{(1-n)it} dt \\
&= i \int_0^{2\pi} (\cos((n-1)t) - i\sin((n-1)t))dt = 0
\end{aligned}
$$

Jordan's Lemma

Assume that for some $R_0 > 0$ the function $g : \mathbb{C} \backslash \overline{B_{R_0}(0)} \to \mathbb{C}$ is ***holomorphic***. Let

$$
f(z) = e^{iaz} g(z), \quad \text{for some } a > 0
$$

Let

$$
C_R = \left\{ z \in \mathbb{C} : z = R \cdot e^{i\theta}, 0 \le \theta \le \pi \right\}
$$

be a ***semi-circle*** segment ***centered at the origin*** in the ***upper half-plane*** and assume that

$$
\sup_{0 \le \theta \le \pi} \left| g\left(Re^{i\theta}\right) \right| \stackrel{R \to \infty}{\longrightarrow} 0
$$

Then

$$
\lim_{R \to \infty} \int_{C_R} f(z) dz = 0
$$

Cauchy Integral Formulas

Suppose f is a ***holomorphic*** function in an open set $\Omega \subset \mathbb{C}$. If D is an open ***disc*** whose boundary is contained in Ω, then

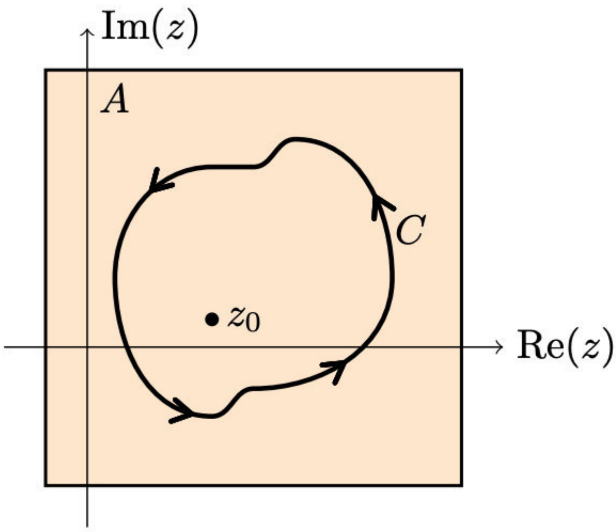
$$
f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} d\zeta \quad \text{for all } z \in D
$$

where $C = \partial D$ is the (***positively oriented***) boundary circle of D.

- The values of a holomorphic function within a disc are fixed by the values of the function on the boundary
- Cauchy's integral formula is also valid for all of our toy contours.

  The reason is actually Cauchy Integral Formulas has a more general way to throw it:

  Suppose $\mathcal{C}$ is a ***simple closed*** curve and the function $f(z)$ is ***holomorphic*** on a region ***containing*** $\mathcal{C}$ and its ***interior***. We assume $\mathcal{C}$ is oriented ***counterclockwise***. Then for any z_0 inside $\mathcal{C}$, the integral formula holds. (How do you understand it?)



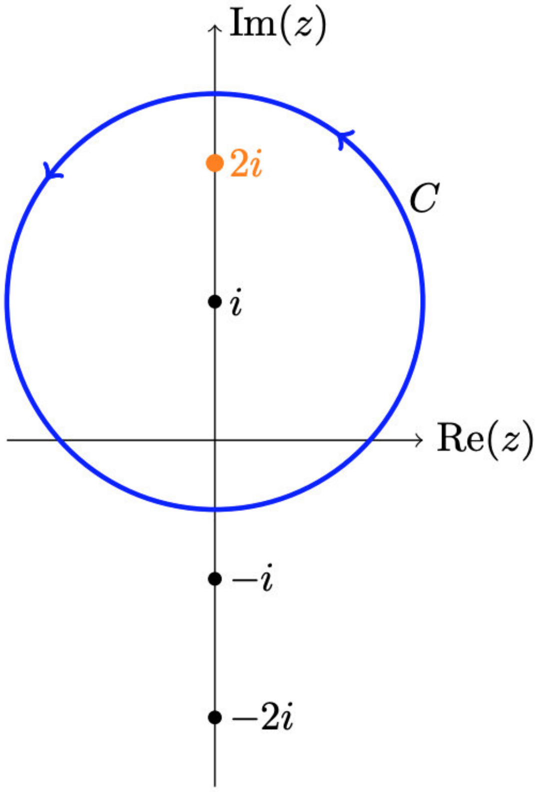
Corollary:

If f is a ***holomorphic*** function in an open set $\Omega \subset \mathbb{C}$, then f has infinitely many complex derivatives in Ω. Moreover, if D is an open disc whose boundary is contained in Ω,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \quad \text{for all } z \in D$$

where $C = \partial D$ is the (***positively oriented***) boundary circle of D.

> **Question**
>
> Compute $\int_C \frac{1}{(z^2+4)^2} dz$ over the contour shown (using Cauchy's integral formula):



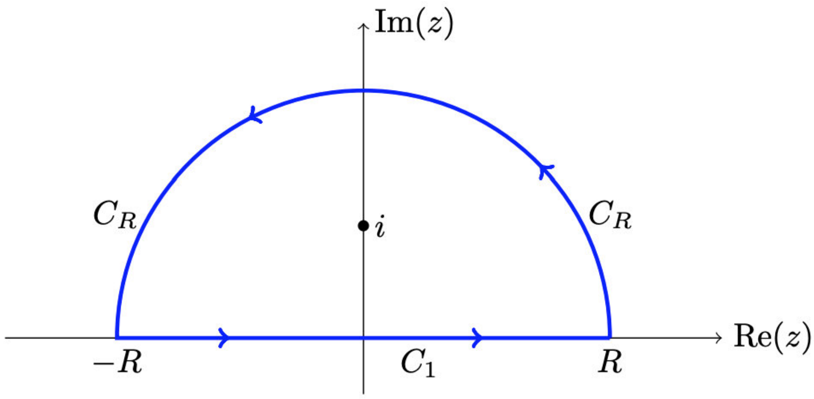
Answer

Evaluate Real Integrations

- ***Extend*** the real domain to complex domain
  - If only containing x, always directly extend to z
  - If containning sinx, cosx, always extend to e^{iz}
- ***Find poles*** for the function $f(z)$
- ***Decide the contour*** and the ***branch*** if needed
  - Semicircle and Indented Semicircle
  - Circle with Keyholes
  - Multiple Kehole
  - Square
- ***Obtain the complex integral*** along the whole contour using theorems or formula
  - **Cauchy's integral theorem** (no poles contained)
  - Cauchy's integral formula (one or two poles, not very complicatied)
  - **The Residue Theorem** (one pole or multiple poles)

- Except for the integral part we need, ***solve or vanish other parts*** one by one
    - May need to use ***Jordan's Lemma*** to prove vanishing
    - May need to use ***triangular inequality*** and ***triangular inequality*** for integrals to prove vanishing

> **Question**
>
> Compute the real integral
>
> $$I = \int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2} dx$$

> **Answer**
>
> 
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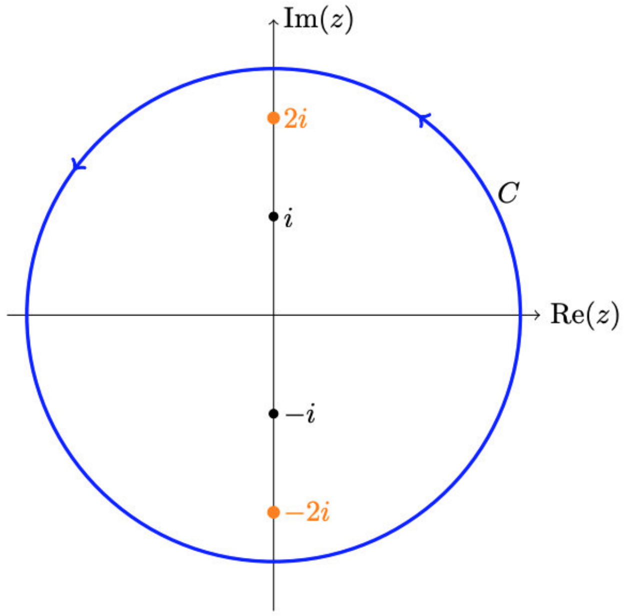
Additional Exercise

> ***Question**
>
> Compute $\int_C \frac{z}{(z^2+4)^2} dz$ over the contour shown (using cauchy's integral formula):



Answer

Hint:

Apply piecewise integration.

And you can use the residue theorem... (coming soon)