VV286 Recitation Class Note

Midterm1 Part1

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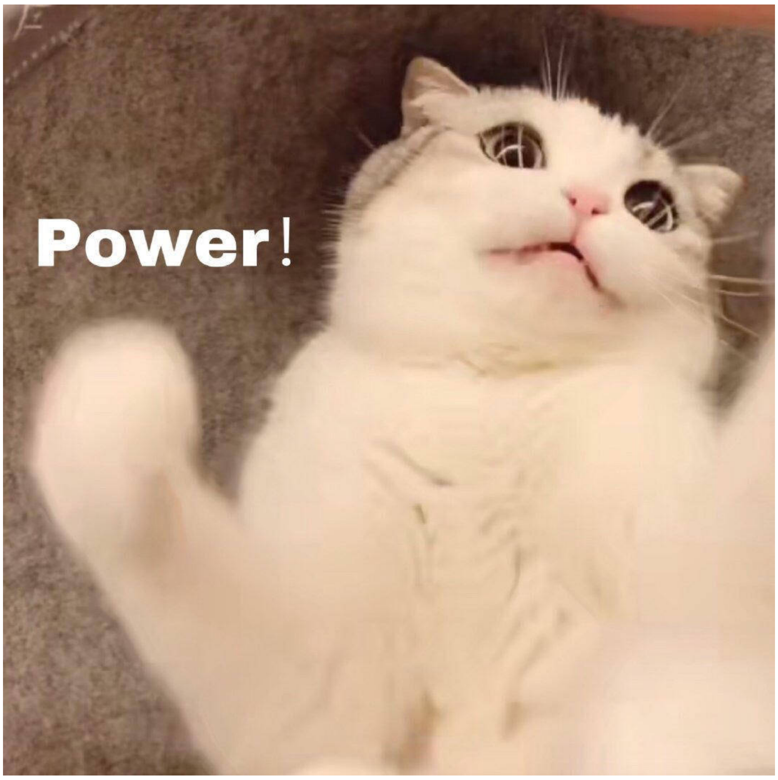


1 Overview

1. Separable Equations
2. Linear Equations
3. Transformable Equations
4. General Integral Curves of First Order ODEs

2 Separable Equations

2.1 Theorem 1.1.3

Let η be an interior point of I_y such that $g(\eta) \neq 0$ and let (Hyp) hold. Then there exists a neighborhood of ξ in I_x in which the IVP

$$y' = f(x)g(y), \quad y(\xi) = \eta$$

has a unique solution y(x). It can be obtained from

$$G(y) = \int_\eta^y \frac{ds}{g(s)} = \int_\xi^x f(t)dt = F(x)$$

by solving for y.

Reminder 1

What is the solution if $g(\eta) = 0$ in Theorem 1.1.3?

1. First of all, you will always have an obvious solution $y(x) = \eta$
2. Second, if $\int_\eta^y \frac{ds}{g(s)}$ exist in a small neighborhood of η, then it's possible to have more solutions, otherwise there's no more solution.

2.2 Equilibrium, Steady-State, Transient Solutions

1. Equilibrium solution:
$$x_{\text{equi}}(t) = \text{ constant}$$
2. Steady-state solution:
$$x_{\text{ss}} = \lim_{t\to\infty} x(t)$$
3. Transient component:
$$x(t) - x_{ss}$$

Exercise 1

Solve the IVP problem:

$$\sqrt{1+4x^2}dy = y^3 x dx, \quad y(0) = -1$$

Solution 1

Separating variables gives us:

$$\frac{dy}{y^3} = \frac{xdx}{\sqrt{1+4x^2}}$$

Integrating gives us:

$$\int \frac{dy}{y^3} = \int \frac{xdx}{\sqrt{1+4x^2}}$$

Let $u = 1 + 4x^2$,

$$\int \frac{dy}{y^3} = \frac{1}{8} \int \frac{du}{\sqrt{u}}$$

So

$$\frac{-1}{2y^2} = \frac{1}{4}\sqrt{1+4x^2} + K$$

Using $y(0) = -1$, we find the solution (since we should find a continuos solution):

$$y = \frac{-\sqrt{2}}{\sqrt{3 - \sqrt{1+4x^2}}}$$

3 Linear Equations

A general linear, first-order ordinary differential equation on an open interval $I \subset \mathbb{R}$ has the form

$$a_1(x)y' + a_0(x)y = f(x), \quad x \in I$$

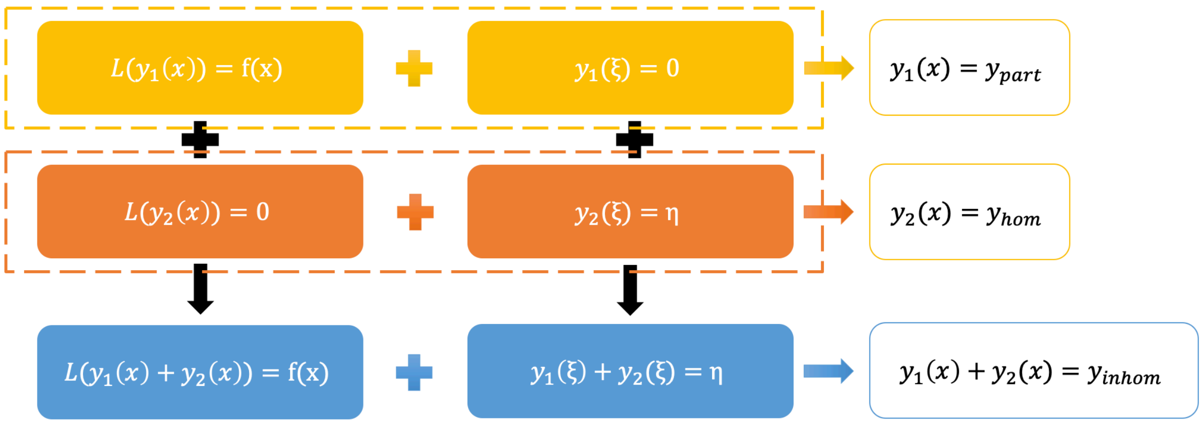
where we allow a_0, a_1, f to be continuous, real-valued functions on I.

3.1 Analyze the Solutions of General IVP

$$a_1(x)y' + a_0(x)y = f(x), \quad x \in I$$
$$y(\xi) = \eta$$

Reminder 2

$L = a_1\frac{d}{dx} + a_2$ represents a linear transformation, which can be viewed as a special operator:...



3.2 Solving the General IVP

3.2.1 Solving y_{hom}

Theorem 1.1.3.

3.2.2 Solving y_{part}

$$a_1(x)y' + a_0(x)y = f(x), \quad x \in I$$
$$y(\xi) = 0$$

1. Duhamel's Principle:
Let $I \subset R$ be an open interval, $x_0 \in \bar{I}$, and a_0,a_1, f continuous, real-valued functions on I, where $a_1(x) \neq 0$ for all $x \in \bar{I}$. Let y_ξ solve the initial value problem

$$a_1(x)y_\xi' + a_0(x)y_\xi = 0, \quad y_\xi(\xi) = \frac{1}{a_1(\xi)}$$

for $x \in \bar{I}$. Then

$$y(x) = \int_{x_0}^{x} f(\xi)y_\xi(x)d\xi$$

solves

$$a_1(x)y' + a_0(x)y = f(x), \quad y\left(x_0\right) = 0$$

2. Variation of Parameters:
Let $y_{part}(x) = c(x)y_{hom}(x)$, then we can solve c(x) from the below equations(why?), and then find $y_{part}(x)$:

$$a_1(x)c'(x)y_{\text{hom}}(x) = f(x)$$
$$c(\xi) = 0$$

3. Integrating Factors:
First solve u(x) from the below equations:

$$u'(x) = \frac{a_0(x)}{a_1(x)}u(x)$$

$$u(\xi) = 1$$

Second solve h(x) = u(x)y(x) from the below equations(why?):

$$h'(x) = \frac{f(x)u(x)}{a_1(x)}$$

$$h(\xi) = \eta$$

Further y(x) = h(x)/u(x).

Exercise 2

Find a general solution for:

$$-\frac{y'}{4} = y + 2$$

Solution 2

This is a special case, and also help you understand the idea behind "integrating factors". Here you can guess out $u = e^{4t}$, let $h = uy = e^{4t}y$, then

$$h' = -8e^{4t}$$

So

$$uy = h = -2e^{4t} + c$$

$$y = ce^{-4t} - 2$$

4 Transformable Equations

1. $y' = f(ax + by + c); b \neq 0$
Let $u(x) = ax + by + c$,
Then $u'(x) = a + bf(u)$.

2. $y' = f(y/x)$
Let $u(x) = \frac{y(x)}{x}$,
Then $u'(x) = (f(u) - u)\frac{1}{x}$.

***3.** $y' = f\left(\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}\right)$
Let $u(x) = a_1x + b_1y(x) + c_1, v(x) = a_2x + b_2y(x) + c_2$,
Then $x = \frac{b_2(u-c_1)-b_1(v-c_2)}{a_1b_2-a_2b_1}$,
And $\frac{du}{dv}$ can be simplified to a form $\frac{du}{dv} = h(u/v)$(how?),
Hence you can solve $u(x) = c(v)$ following 5.2,
And then obtain the relationship between x and y.

4. $y' + gy + hy^\alpha = 0, \alpha \neq 1$ **(Bernoulli's equation)**
Let $u(x) = y^{1-\alpha}$,
Multiply both side with u(x),
then $u'(x) + (1-\alpha)g(x)u(x) + (1-\alpha)h(x) = 0$.

5. $y' + gy + hy^2 = k$ **(Ricatti's equation)**
First you guess/know a solution $\phi(x)$,
Let $u(x) = y(x) - \phi(x)$,
From

$$\begin{cases} y' + gy + hy^2 = k \\ \phi' + g\phi + h\phi^2 = k \end{cases}$$

,
You obtain $u' + gu + h\left(y^2 - \phi^2\right) = 0$, which gives $u' + (g + 2\phi h)u + hu^2 = 0$.
Solve this Bernoulli's equation where $\alpha = 2$.

Exercise 3

Find every nonzero solution of the differential equation

$$y' = y + 2y^5$$

Solution 3

This is a Bernoulli equation for $\alpha = 5$. Multiply the equation by $u = y^{-4}$, then

$$-\frac{u'}{4} = u + 2$$

Which is the function we solved in exercise 2.

$$u = ce^{-4t} - 2$$

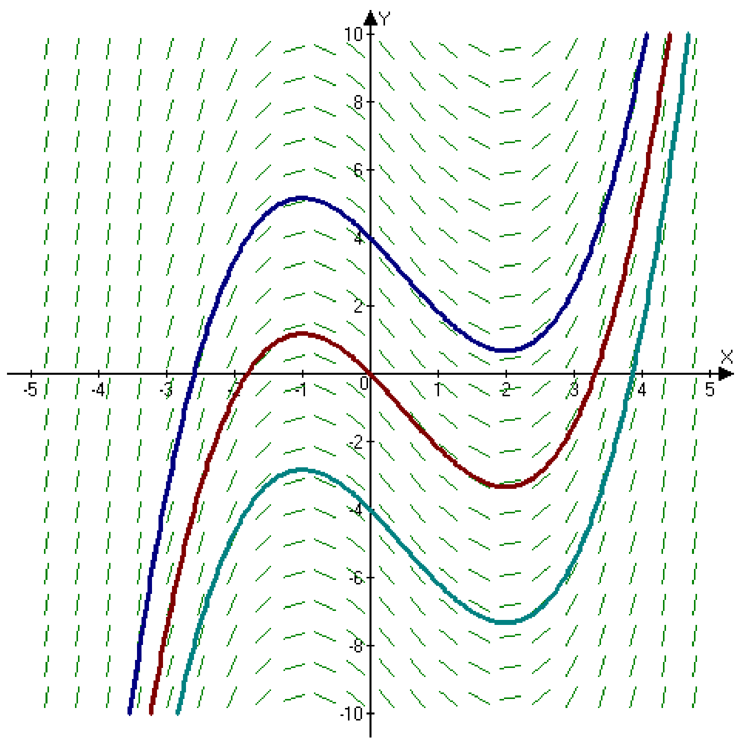
$$y(t) = \pm\frac{1}{(ce^{-4t} - 2)^{1/4}}$$

5 General Integral Curves of First Order ODEs

5.1 Overview

$$h(x,y)y' + g(x,y) = 0$$
$$y' = -\frac{g(x,y)}{h(x,y)}, \quad if\, h \neq 0$$

1. A vector field F(x,y) and integral curves for F(x,y)?
2. An integral curve describes a trajectory, and a function for a trajectory describes the positions and thus describes the relationship between x and y, so it can be seen as a solution to certain functions. Besides, y' is the slope, containing the direction of vectors (-h(x,y),g(x,y)) in F(x,y) and (g(x,y),h(x,y)) in $F^{\perp}(x,y)$.
3. If for $F^{\perp}(x,y)$, $U^{\perp}(x,y) = C_0$ exists, then integral curves for F(x,y) is just trajectories where $U^{\perp}(x,y) = C_0$ is constant.
4. If initially $F^{\perp}(x,y)$ do not have $U^{\perp}(x,y)$, we try to adjust the fields a little without changing the directions.



5.2 Solving ODE with an Integral Curve

1. Judge $\begin{pmatrix} g(x,y) \\ h(x,y) \end{pmatrix}$ whether: $\frac{\partial g(x,y)}{\partial y} = \frac{\partial h(x,y)}{\partial x}$
2. If not, $\begin{pmatrix} M(x,y)g(x,y) \\ M(x,y)h(x,y) \end{pmatrix}$, find M so that $\frac{\partial (Mg)}{\partial y} = \frac{\partial (Mh)}{\partial x}$.

   i Most general: $M_y g + M g_y = M_x h + M h_x$

   ii Let $M(x,y) = M(x)$: $M'(x)h = M(g_y - h_x) \Rightarrow \frac{M'(x)}{M(x)} = \frac{g_y - h_x}{h} = (\ln M(x))'$

3. Find $U^{\perp}$ for $F^{\perp} = \begin{pmatrix} Mg \\ Mh \end{pmatrix}$.

   i $\frac{\partial U^t}{\partial x} = Mg \Rightarrow U^{\perp} = \int Mgdx + f(y)$
   $\frac{\partial (\int Mgdx)}{\partial y} + f'(y) = Mh \Rightarrow get\, f(y)$
   $\Rightarrow get\, U^{\perp}(t)$

   ii $\begin{cases} \frac{\partial U^{\perp}}{\partial x} = Mg \\ \frac{\partial U^{\perp}}{\partial y} = Mh \end{cases} \Rightarrow \begin{cases} U^{\perp} = \int Mgdx + f(y) \\ U^{\perp} = \int Mhdy + k(x) \end{cases} \Rightarrow U^{\perp}$

4. The solution is in the form $U^{\perp}(x,y) = C_0$.

Exercise 4

Find all solutions y to the differential equation

$$\left(t^2 + ty\right) y' + \left(3ty + y^2\right) = 0$$

Solution 4

We first verify whether this equation is exact(i.e. $U^{\perp}(x,y)$) exist.

$$\begin{aligned} h(t,y) &= t^2+ty &\Rightarrow\quad \partial_t h(t,y) = 2t+y \\ g(t,y) &= 3ty+y^2 &\Rightarrow\quad \partial_y g(t,y) = 3t+2y, \end{aligned}$$

The differential equation is not exact. Let

$$\begin{aligned} \frac{M'}{M} &= \frac{\partial_y g(t,y)-\partial_t h(t,y)}{h(t,y)} \\ &= \frac{(3t+2y)-(2t+y)}{(t^2+ty)} \\ &= \frac{(t+y)}{t(t+y)} \\ &= \frac{1}{t} \quad \Rightarrow \quad \frac{M'}{M} = \frac{1}{t} \end{aligned}$$

Solving it gives $M(t)=t$.

$$\begin{aligned} \tilde{h}(t,y) &= t^3+t^2y \\ \tilde{g}(t,y) &= 3t^2y+ty^2 \end{aligned}$$

Then

$$\partial_y U^{\perp} = t^3+t^2y \Rightarrow U^{\perp}(t,y) = \int \left(t^3+t^2y\right) dy + f(t)$$

So

$$U^{\perp}(t,y) = t^3y+\frac{1}{2}t^2y^2+f(t)$$

And

$$3t^2y+ty^2+f'(t) = \partial_t U^{\perp}(t,y) = \tilde{g}(t,y) = 3t^2y+ty^2$$

So let $f(t)=0$. $U^{\perp}(t,y)=t^3y+\frac{1}{2}t^2y^2$.
All solutions y to the differential equation satisfy the equation

$$t^3y(t)+\frac{1}{2}t^2(y(t))^2 = c_0$$