VE401 Recitation Class Note9

Single Sample Test

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1 Parametric Test for Mean

Distribution of X_i	Sample size n	Variance σ^2	Statistic
$X_i \sim \mathcal{N}(\mu, \sigma)$	any	known	$\frac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim \mathcal{N}(0,1)$
$X_i \sim$ any distribution	large	known	$\frac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim \mathcal{N}(0,1)$
$X_i \sim$ any distribution	large	unknown	$\frac{\overline{X}-\mu}{\frac{s}{\sqrt{n}}} \sim \mathcal{N}(0,1)$
$X_i \sim \mathcal{N}(\mu, \sigma)$	small	unknown	$\frac{\overline{X}-\mu}{\frac{s}{\sqrt{n}}} \sim t_{n-1}$
$X_i \sim$ any distribution	small	known or unknown	Go home!

1.1 Z-test

Test Statistic:

$$Z = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}$$

Reject at significance level α:

(i) $H_0 : \mu = \mu_0$ if $|Z| > z_{\alpha/2}$

(ii) $H_0 : \mu \leq \mu_0$ if $Z > z_{\alpha}$

(iii) $H_0 : \mu \geq \mu_0$ if $Z < z_{-\alpha}$

Abscissa of OC Curves:

$$d = \frac{|\mu - \mu_0|}{\sigma}$$

1.2 T-test

Test Statistic:

$$T_{n-1} = \frac{\overline{X} - \mu_0}{S/\sqrt{n}}$$

Reject at significance level α:

(i) $H_0 : \mu = \mu_0$ if $|T_{n-1}| > t_{\alpha/2}$

(ii) $H_0 : \mu \leq \mu_0$ if $T_{n-1} > t_\alpha$

(iii) $H_0 : \mu \geq \mu_0$ if $T_{n-1} < t_{-\alpha}$

Abscissa of OC Curves:

$$d = \frac{|\mu - \mu_0|}{\sigma}$$

Estimate σ:

(i) Use prior experiments to insert a rough estimate for σ

(ii) Express the difference $\delta = |\mu - \mu_0|$ relative to σ

(iii) Substitute the sample standard deviation s for σ.

2 Parametric Test for Variance

Distribution of X_i	Sample size n	Mean μ	Statistic
$X_i \sim \mathcal{N}(\mu, \sigma)$	any	known or unknown	$\frac{s^2(n-1)}{\sigma^2} \sim \chi^2_{n-1}$

2.1 Chi-squared Test

Test Statistic:

$$\chi^2_{n-1} = \frac{(n-1)S^2}{\sigma_0^2}$$

Reject at significance level α:

(i) $H_0 : \sigma = \sigma_0$ if $\chi^2_{n-1} > \chi^2_{\alpha/2,n-1}$ or $\chi^2_{n-1} < \chi^2_{1-\alpha/2,n-1}$

(ii) $H_0 : \sigma \leq \sigma_0$ if $\chi^2_{n-1} > \chi^2_{\alpha,n-1}$

(iii) $H_0 : \sigma \geq \sigma_0$ if $\chi^2_{n-1} < \chi^2_{1-\alpha,n-1}$

Abscissa of OC Curves:

$$d = \frac{\sigma}{\sigma_0}$$

3 Non-parametric Test for Median

1. Non-parametric statistics: do not assume the dependence on any parameter.
2. Distribution-free statistics: do not assume that X follows any particular distribution (such as the nor- mal distribution).

3.1 Sign Test

Number of Signs:

Let $X_1, \ldots, X_n$ be a random sample of size n from an arbitrary continuous distribution and let

$$Q_+ = \#\left\{X_k : X_k - M_0 > 0\right\}, \quad Q_- = \#\left\{X_k : X_k - M_0 < 0\right\}$$

Q_+ is the number of "positive signs" and Q the number of "negative signs".

If $X_i - M_0 = 0$, usual practice is to exclude X_i from the analysis.

Calculating P-value:

$$P\left[Q_- \leq k \mid M = M_0\right] = \sum_{x=0}^{k} \binom{n}{x} \frac{1}{2^n}$$

Rejecting H_0:

We reject at significance level α:

(i) $H_0 : M \leq M_0$ if $P\left[Q_- < k \mid M = M_0\right] < \alpha$

(ii) $H_0 : M \geq M_0$ if $P\left[Q_+ < k \mid M = M_0\right] < \alpha$

(iii) $H_0 : M = M_0$ if $P\left[\min\left(Q_-, Q_+\right) < k \mid M = M_0\right] < \alpha/2$

3.2 Wilcoxon Signed Rank Test

Sum of Ranks:

Let $X_1, \ldots, X_n$ be a random sample of size n from a symmetric distribution.

Order the n absolute differences $|X_i - M_0|$ according to magnitude, so that $X_{R_i} - M_0$ is the ${R_i}^t h$ smallest difference by modulus.

If ties in the rank occur, the mean of the ranks is assigned to all equal values.

Let

$$W_+ = \sum_{R_i > 0} R_i, \quad |W_-| = \sum_{R_i < 0} |R_i|$$

W_+ is the sum of "positive ranks" and W the modulus of the sum of "negative signs".

If $X_i - M_0 = 0$, usual practice is to exclude X_i from the analysis.

The Critical Value:

For small n, for example ≤ 30, critical values are given in tables.

Rejecting H_0:

We reject at significance level α:

(i) $H_0 : M \leq M_0$ if W_- is smaller than the critical value for α

(ii) $H_0 : M \geq M_0$ if W_+ is smaller than the critical value for α

(iii) $H_0 : M = M_0$ if $W = min(W_+, |W_-|)$ is smaller than the critical value for $\alpha/2$

Calculating P-Value:

For large n, we use the normal distribution to approximately calculate P-value, with parameters

$$\mu = \frac{n(n+1)}{4}, \quad \sigma^2 = \frac{n(n+1)(2n+1)}{24}$$

Rejecting H_0:

We reject at the [P-value] significance level if [P-value] is small.

4 Inferences on Proportion

4.1 Estimator and Statistic

Define the random variable:

$$X = \begin{cases} 1 & \text{has trait,} \\ 0 & \text{does not have trait.} \end{cases}$$

If we take a random sample X_1,...,X_n of X, the sample mean is an (unbiased) estimator for p:

$$\widehat{p} = \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

By the central limit theorem, when n is large enough, $\widehat{p}$ is approximately normally distributed with mean p and variance p(1 − p)/n. Hence,

$$Z = \frac{\widehat{p} - p}{\sqrt{p(1-p)/n}}$$

4.2 Interval Estimation and Sample Size

It follows immediately that the following is a 100(1 − α)% confidence interval for p:

$$\widehat{p} \pm z_{\alpha/2}\sqrt{p(1-p)/n}$$

But the interval depends on the unknown parameter p, which we are actually trying to estimate! One solution to the problem is to replace p by $\widehat{p}$, i.e., to write

$$\widehat{p} \pm z_{\alpha/2}\sqrt{\widehat{p}(1-\widehat{p})/n}$$

We may want to be able to claim that "with 100(1 − α)% probability, $\widehat{p}$ differs from p by at most d."

Given a 100(1 − α)% confidence interval p = $\widehat{p}$ ± z know with 100(1 − α)% confidence that

$$d = z_{\alpha/2}\sqrt{\widehat{p}(1-\widehat{p})/n}$$

Given d, this means that we should choose

$$n = \frac{z_{\alpha/2}^2\widehat{p}(1-\widehat{p})}{d^2}$$

which requires us to have an estimate $\widehat{p}$ of p beforehand.

4.3 Test for Proportion

Let X_1, ... , X_n be a random sample of (large) size n from a Bernoulli distribution with parameter p and let $\widehat{p}$ = X denote the sample mean. Then any test based on the statistic

$$Z = \frac{\widehat{p} - p_0}{\sqrt{p_0\left(1 - p_0\right)/n}}$$

is called a large-sample test for proportion.
We reject at significance level α:

(i) $H_0 : p = p_0$ if $|Z| > z_{\alpha/2}$

(ii) $H_0 : p \leq p_0$ if $Z > z_\alpha$

(iii) $H_0 : p \geq p_0$ if $Z < z_\alpha$

5 Acceptance Sampling

Acceptance Sampling

A construction firm receives a shipment of N = 50 steel rods to be used in the construction of a bridge. The lot must be checked to ensure that the breaking strength of the rods meets specifications. The lot will be rejected if among the 200 rods more than 10% fail to meet specifications. We define the true proportion of defective for the 200 rods as Π. We test:

$$H_0 : \Pi \leq 0.1, \qquad H_1 : \Pi - 0.1 > \delta = 0.1$$

We test a sample of size n=20 and reject H_0 if more than c=3 rod fails to meet specifications.

1. What is α?
2. What is β?
3. *Can you draw the OC curve?
4. Keep α and n the same, if we want to make $\beta = 0.2$, what H_1 will you set?
5. To make $\alpha = 0.05$, assume n = 10, what value of c will you choose?
6. **To make $\alpha = 0.05$, $\beta = 0.3$, what value of n will you choose?