VE401 Recitation Class Note5

Transformation of Random Variables and Reliability

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1 Transformation of Random Variables

Theorem:
Let (X , f_X) be a continuous multivariate random variable and let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable, bijective map with inverse φ^{-1}. Then Y = $\phi \circ X$ is a continuous multivariate random variable with density:

$$f_Y(y) = f_X \circ \varphi^{-1}(y) \cdot \left|\det D\varphi^{-1}(y)\right|$$

where $D\varphi^1$ is the Jacobian of φ^1:

$$\boldsymbol{D}\varphi^{-1} = \begin{pmatrix} \frac{\partial\varphi_1^{-1}}{\partial y_1} & \frac{\partial\varphi_1^{-1}}{\partial y_2} & \cdots & \frac{\partial\varphi_1^{-1}}{\partial y_n} \\ \frac{\partial\varphi_2^{-1}}{\partial y_1} & \frac{\partial\varphi_2^{-1}}{\partial y_2} & & \frac{\partial\varphi_2^{-1}}{\partial y_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial\varphi_n^{-1}}{\partial y_1} & \frac{\partial\varphi_n^{-1}}{\partial y_2} & \cdots & \frac{\partial\varphi_n^{-1}}{\partial y_n} \end{pmatrix}$$

Comments:

1. Notice this theorem is for differentiable and bijective φ
2. Many other types of transformation can be calculated based on this
3. We will also discuss later interesting transformation can not use this theorem

1.1 Bivariate Random Variables

First recall in class, we have proved:

Theorem:
Let ((X , Y), f_{XY}) be a continuous bivariate random variable. Let U = X /Y . Then the density f_U of U is given by:

$$f_U(u) = \int_{-\infty}^{\infty} f_{XY}(uv, v) \cdot |v| dv$$

Comments:

1. It's important that you know how to prove.
2. The general idea is you first define $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ to get a joint density function of f_{UV}, and then calculate the marginal density for U.

Next let's see more cases.

Question: Transform Bivariate Random Variables

Let ((X , Y), f_{XY}) be a continuous bivariate random variable. Calculate the density f_U of U when:

1. U = X + Y
2. U = XY

Answer: Transform Bivariate Random Variables

1. U = X + Y:
$$f_U(u) = \int_{-\infty}^{\infty} f_{XY}(u - v, v)dv$$
2. U = XY
$$f_U(u) = \int_{-\infty}^{\infty} f_{XY}(\frac{u}{v}, v) \cdot |\frac{1}{v}|dv$$

Question: Further discussion when U = X + Y

1. When X and Y are independent, further simplify your result for $f_U(u)$, U = X + Y.
2. When X and Y are independent, U = X + Y, it is given that $m_U(t) = m_X(t)m_Y(t)$. (If you are interested in the proof, you can discuss with me later.)

   Can you use this to prove that:

   A: The sum of two i.i.d exponential distribution random variables is gamma distributed.

   B: The sum of two independent gamma distribution random variables with the **same parameter** β is still gamma distributed with parameters $(\alpha_1 + \alpha_2, \beta)$.

Answer: Further discussion when U = X + Y

1. $f_U(u) = \int_{-\infty}^{\infty} f_Y(u-v)f_X(v)dv = f_X * f_Y$.
2. First calculate $m_U(t) = m_X(t)m_Y(t)$; second by the uniqueness of MGF, you can read out the distribution of U.

   Do notice there are restrictions in the problem:

   A: X and Y are two i.i.d exponential distribution random variables.

$$m_X(t) = \left(1 - \frac{t}{\beta}\right)^{-1}$$
$$m_Y(t) = \left(1 - \frac{t}{\beta}\right)^{-1}$$
$$m_U(t) = \left(1 - \frac{t}{\beta}\right)^{-2}$$

   B: Two independent gamma distribution random variables with the **same parameter** β.

$$m_X(t) = \left(1 - \frac{t}{\beta}\right)^{-\alpha_1}$$
$$m_Y(t) = \left(1 - \frac{t}{\beta}\right)^{-\alpha_2}$$
$$m_U(t) = \left(1 - \frac{t}{\beta}\right)^{-\alpha_1-\alpha_2}$$

1.2 Chi Distribution χ_n

Definition:

$$\chi_n := \sqrt{\sum_{i=1}^{n} Z_i^2}$$

$$f_{\chi_n}(y) = \frac{2}{2^{n/2}\Gamma\left(\frac{n}{2}\right)} y^{n-1} e^{-y^2/2}$$

Interpretation:

Suppose all Z_i are independent and follow the standard normal distribution. $z = (z_1, ..., z_n)$ is a point's position. Then the distance of a point from the origin follows a χ_n distribution.

1.3 *Discussion: Imagine χ_n and χ_n^2 Distribution

Let's discuss and understand the χ_n and χ_n^2 Distribution from the below graph.

1. First understand the independence of X and Y

   Hint: 1. You can understand using our previous introduction of bivariate normal distribution and this 3D graph; 2. You can also understand in a general sense of independence graphically.

2. Second understand the Chi variable $\sqrt{X^2 + Y^2}$

3. Third understand the PDF f_{χ_n}

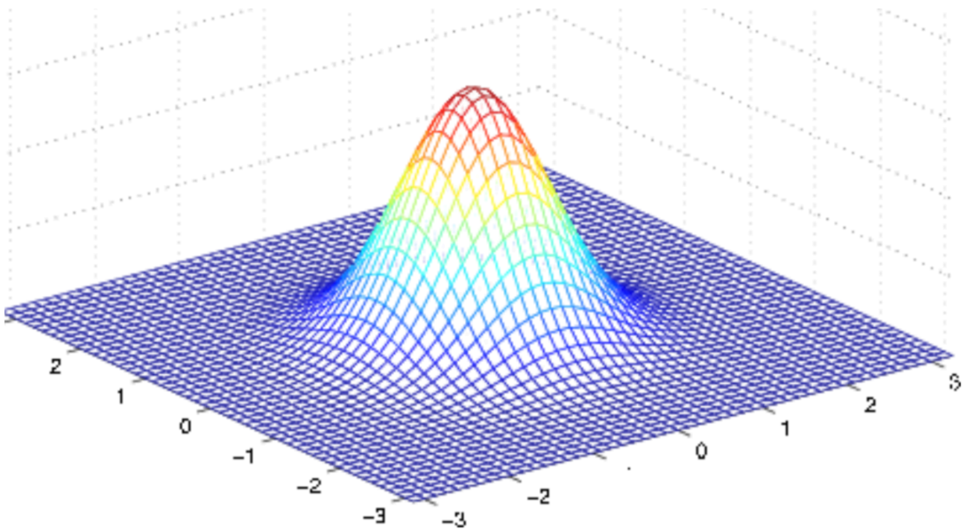


Figure 1: Imagine the Chi Distribution

And the $\chi_n^2 = \sum_{i=1}^{n} Z_i^2$ has similar interpretations.

2 Reliability

2.1 A Single Unit

The time for A to fail is described as a continuous random variable T_A.

Failure Density:
The probability density function of T_A is called the failure density f_A.
Then we can also define a CDF F_A.

Reliability Function:
The probability that A is still working at time t is described with the reliability function:

$$R_A(t) = 1 - F_A(0)$$

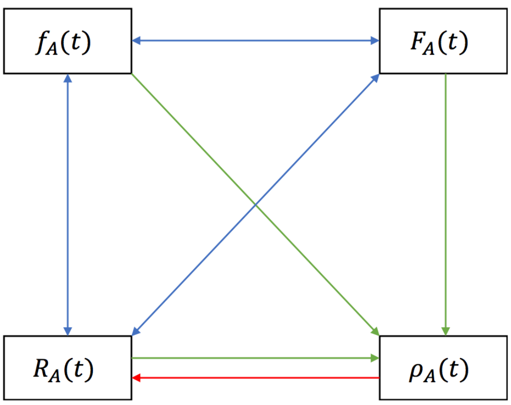
Hazard Rate:
The hazard rate is defined as:

$$\begin{aligned} \varrho_A(t) &= \lim_{\Delta t \to 0} \frac{P[t \leq T \leq t + \Delta t | t \leq T]}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{P[t \leq T \leq t + \Delta t]}{P[T \geq t] \cdot \Delta t} \\ &= f_A(t)/R_A(t) \end{aligned}$$

Interpret Hazard Rate:
Given unit A keeps working before time t, the failure "density" for A at time t.
$\varrho_A(t)$ decreasing: As time goes by, a working unit will be less likely to occur.
$\varrho_A(t)$ steady: As time goes by, a working unit will be equally likely to occur.
$\varrho_A(t)$ increasing: As time goes by, a working unit will be more likely to occur.
Property:

$$R(t) = e^{-\int_0^t \varrho(x) dx}$$

*Draw all the equations relating the four in the below figure:



2.2 Weibull Distribution

PDF:

$$f(x) = \begin{cases} \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}}, & x > 0 \\ 0, & \text{otherwise} \end{cases} \quad \alpha, \beta > 0$$

Features:

1. $\mathrm{E}[X] = \alpha^{-1/\beta}\Gamma(1 + 1/\beta)$
2. $\mathrm{Var}\, X = \alpha^{-2/\beta}\Gamma(1 + 2/\beta) - \mu^2$

If f_A follows Weibull Distribution:

1. $\varrho_A(t) = \alpha\beta t^{\beta-1}$
2. $R_A(t) = e^{-\alpha t^{\beta}}$

Comments:

When $\beta = 1$, f_X becomes exactly the PDF for an exponential distributed random variable.
The exponential distribution is memoryless.
Also when $\beta = 1$, $\varrho_A(t)$ is constant.
Now you can link them together. Recall that we have discussed a machine won't "remember" it has worked for some period in a precious recitation class.

2.3 A System

A series system with k components:

$$R_s(t) = \prod_{i=1}^{k} R_i(t)$$

A parallel system with k components:

$$R_p(t) = 1 - P[\text{ all components fail before } t] = 1 - \prod_{i=1}^{k} (1 - R_i(t))$$

2.4 *Discussion: System & Transforming Multivariate RVs

Question: A System

Let S be a series system of two components A and B.
It is known that component the failure density of A and B are both exponential distributed with parameter β_1 and β_2.
Then:

1. What is the reliability function of the system S?
2. Observe $R_S(t)$, what is the the failure density of S?

Answer: A System

1. $R_S(t) = e^{-(\beta_1+\beta_2)t}$
2. $f_S(t) = -R_S'(t) = (\beta_1 + \beta_2)e^{-(\beta_1+\beta_2)t}$
   Which follows the exponential distribution.

Question: System-Transformation of Variables 1

X_1 and X_2 are two independent random variables following the exponential distribution with parameter β_1 and β_2. Y = $\min\{X_1, X_2\}$.
What is the PDF $f_Y(y)$?

Answer: System-Transformation of Variables 1

Using the idea of series system. Define three variables.
X_1 the fail time of component A,
X_2 the fail time of component B,
Y the fail time of system S.
When any component fails, the system will fail. So Y = $\min\{X_1, X_2\}$.
Based on the previous question, we have:
$f_Y(y) = (\beta_1 + \beta_2)e^{-(\beta_1+\beta_2)y}$

Question: System-Transformation of Variables 2

For random variables Y, X_1, and X_2, how will you describe the transformation using a system if:

1. Y = $X_1 + X_2$
2. Y = $\max\{X_1, X_2\}$

You can draw it.

Answer: System-Transformation of Variables 2

1. Y = $X_1 + X_2$: A back-up system (with a switch).
2. Y = $\max\{X_1, X_2\}$: A parallel system.