# VE401 Recitation Class Note1

## Introduction to Probability

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# 1 About Me

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# 2 Elementary Probability

## 2.1 Cardano's Principle

Let A be a random outcome of an experiment that may proceed in various ways. Assume each of these ways is **equally likely.** Then the probability P[A] of the outcome A is:

$$P[A] = \frac{number\, of\, ways\, leading\, to\, outcome\, A}{total\, number\, of\, ways\, (the\, experiment\, can\, proceed)}$$

**Example1: Rolling Two Coins**

Total 4 ways: (h,h), (h,t), (t,h), (t,t),
Event X–At least 1 head: (h,h), (h,t), (t,h). So P[X] = $\frac{3}{4}$
• Unequally likely ways–D'Alembert's Error
• Indistinguishable coins–Bose's inspiration

## 2.2 Counting Ways for Events

Suppose a set A of n objects is given. From A:

1. **choose k ordered objects:** $\frac{n!}{(n-k)!}$
2. **choose k unordered objects:** $\frac{n!}{k!(n-k)!}$
3. **partitioning A into k disjoint subsets** $A_1, ..., A_k$ **having** $n_1, ..., n_k$ **elements:** $\frac{n!}{n_1!...n_k!}$

Understand: Order the total n objects, use clapboard to separate, consider repetition.

**Question1: Counting**

(i) An NBA basketball team has 12 players. A starting team consists of 5 players. How many possible ways to form a starting team?

(ii) Given 1 "T", 4 "E", 2 "N", and 2 "S", how many different words can you form?

## 2.3 Axiomatic Approach

### 2.3.1 $\sigma-$Field

S is a non-empty set. A $\sigma-$field $\mathcal{F}$ on S is a set of subsets of S such that:

(i) $\emptyset \in \mathscr{F}$

(ii) if $A \in \mathscr{F}$, then $S \backslash A \in \mathscr{F}$

(iii) if $A_1, A_2, ... \in \mathscr{F}$ is a finite or countable sequence of subsets, then the union $\bigcup_k A_k \in \mathscr{F}$.

### 2.3.2 Probability Measures and Spaces

Let S be a sample space and $\mathscr{F}$ a $\sigma-$field on S. A **probability measure** (or **probability function** or just **probability**) on S is a **function** such that:

$$P : \mathscr{F} \to [0, 1], \quad A \mapsto P[A]$$

(i) P[S]=1

(ii) For any set of events $A_k \subset \mathscr{F}$ such that $A_j \cap A_k = \emptyset$ for $j \neq k$ ,

$$P\left[\bigcup_k A_k\right] = \sum_k P[A_k]$$

Then $(S, \mathscr{F}, P)$ is called a **probability space**.

### 2.3.3 Almost Surely

A $\in \sigma-$field, and P[A] = 1. This does not means A = S.

**Example2: Stop Until Heads appear**

$$S = \{h, th, thh, ...\} \cup \{t^\infty\}$$

$$A = \{h, th, thh, ...\}$$

$$P[\{\underbrace{t \cdots t}_{n \text{ times}} h\}] = \frac{1}{2^{n+1}}, \; P[A] = 1$$

#### 2.3.4 Properties of Probability

$$\mathrm{P[S]} = 1$$
$$\mathrm{P}[\emptyset] = 0$$
$$\mathrm{P[S \setminus A]} = 1 - \mathrm{P[A]}$$
$$\mathrm{P}[A_1 \cup A_2] = \mathrm{P}[A_1] + \mathrm{P}[A_2] - \mathrm{P}[A_1 \cap A_2]$$

# 3 Conditional Probability

## 3.1 Definition

Given event A occurs, the probability for B to occur:

$$P[B|A] := \frac{P[A \cap B]}{P[A]}$$

**Example3: Two Children Problem**

A mother has two children, one is a girl, What is the probability that the other child is a boy? To avoid ambiguity:

(i) The mother has a daughter.

(ii) The mother's older child is a girl.

Other comments:

(i) Natural language may cause weird ambiguity.

(ii) Various models described by natural language come from a basic model. Sometimes the one create an extended model do not notice the possible ambiguity under the surface. In most times for you to deal, go back to the simplest model is preferred.

## 3.2 Independence

Event A and event B are independent if:

$$P[A \cap B] = P[A] \cdot P[B]$$

And then:

$$P[B|A] = P[B],\ if\ P[A] \neq 0$$
$$P[A|B] = P[A],\ if\ P[B] \neq 0$$

**Question2: Mutually Exclusive and Independence**

Are mutually exclusive and Independence the same?
Of course not. Many examples. Can anyone give one?
Besides, try to fill the table below:

| Relationship between A, B | Mutually exclusive | Independent |
|---|---|---|
| P[A∩B] | 0 | P[A]·P[B] |
| P[A∪B] | | |
| P[A\|B] | | |
| P[A\|¬ B] | | |
| P[¬A\|B] | | |

## 3.3 Total Probability

$$P[B] = \sum_{k=1}^{n} P\left[B|A_k\right] \cdot P\left[A_k\right]$$

**Example4: Marriadge Problem**

Optimal strategy: For some r  1, evaluate and automatically reject r − 1 potential partners. Then select the first candidate superior to all the previous ones, if possible.
Sample space: S = {(k,j): k is selected and $x_j$ = 1, k = 0,...,n, j = 1,...,n}, k=0 means no one is selected.
Event–winning the best: $W_r$ = {(k,k): k = r,...,n}, $r \geqslant 1$.
Event–make choice when the mth person is the best: $B_m$ = {(k,m): k = 0,...,n }, m = 1,...,n. So P[$B_m$]=$\frac{1}{n}$.
Consider $P\left[W_r|B_m\right]$: Need the best candidate of the first m-1 candidates occurs in the directly rejected first r-1 candidates. So $P\left[W_r|B_m\right] = \frac{r-1}{m-1}$.
Probability of winning:

$$P\left[W_r\right] = \sum_{m=1}^{n} P\left[W_r|B_m\right] P\left[B_m\right] = \frac{1}{n} \sum_{m=r}^{n} \frac{r-1}{m-1}$$

Then simplify using $\sum_{k=1}^{n} \frac{1}{k} \approx \ln(n) - \gamma$. r = $\lceil \frac{n}{e} \rceil$, $P_{max} \approx \frac{1}{e}$.

## 3.4 Bayes's Theorem

$A_1, ..., A_n \subset S$ and pairwise mutually exclusive;
$\bigcup_n A_n = S$;

$B \subset S$ and $P[B] \neq 0$. Then:

$$P\left[A_k|B\right] = \frac{P\left[B \cap A_k\right]}{P[B]} = \frac{P\left[B|A_k\right] \cdot P\left[A_k\right]}{\sum_{j=1}^{n} P\left[B|A_j\right] \cdot P\left[A_j\right]}$$

**Example5: Monty Hall Paradox**

You have chosen door A. Then door C is shown to harbor a goat. To win the money, should you switch to door B?
Yes. Win money from B, equals to your initial choice is wrong, so equals to $1 - \frac{1}{3} = \frac{2}{3}$.
Mathematically:

$$\begin{aligned} P\left[A|C^*\right] &= \frac{P\left[C^*|A\right] \cdot P[A]}{P\left[C^*|A\right] \cdot P[A] + P\left[C^*|C\right] \cdot P[C] + P\left[C^*|B\right] \cdot P[B]} \\ &= \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{2} \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3}} = \frac{1}{3} \end{aligned}$$

**Question3: A Machine**

There is a machine. When the machine is well-adjusted, the probability to produce a qualified product is 0.98. When the machine is not well-adjusted, the probability to produce a qualified product is 0.55. Every morning, when the machine is turned on, the probability for it to be adjusted well is 0.95. One day, after turning on the machine, the first produced product is qualified. Then, what is the probability that the machine is adjusted well?

# 4 Answers

**Answer1: Counting**

(i) $\frac{12!}{5!\cdot 7!} = 792$

(ii) Think: Total 9 positions. Need to partitioning 9 positions into 4 subgroups, each containing 1, 4, 2, and 2 positions. $\frac{9!}{1!\cdot 4!\cdot 2!\cdot 2!} = 3780$

**Answer2: Mutually Exclusive and Independence**

| Relationship between A, B | Mutually exclusive | Independent |
|---|---|---|
| P[A∩B] | 0 | P[A]·P[B] |
| P[A∪B] | P[A]+P[B] | P[A]+P[B]-P[A]·P[B] |
| P[A\|B] | 0 | P[A] |
| P[A\|¬ B] | $\frac{P[A]}{1-P[B]}$ | P[A] |
| P[¬A\|B] | 1 | 1-P[A] |

**Answer3: A Machine**

Event A: the product is qualified.
Event B: the machine is well adjusted.
P[A|B]=0.98, P[A|$\neg B$]=0.55, P[B]=0.95, P[$\neg B$]=0.05.

$$P[B|A] = \frac{P[A|B]\cdot P[B]}{P[A|B]\cdot P[B] + P[A|\neg B]\cdot P[\neg B]} = \frac{0.98\times 0.95}{0.98\times 0.95 + 0.55\times 0.05} = 0.97$$